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Distributed discrete-time coordinated tracking with Markovian switching topologies

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ABSTRACT

This paper deals with the distributed discrete-time coordinated tracking problem for multi-agent systems with Markovian switching topologies. In the multi-agent team, only some of the agents can obtain the leader's state directly. The leader's state considered is time varying. We present necessary and sufficient conditions for boundedness of the tracking error system and show the ultimate bound of the tracking errors. A linear matrix inequality approach is developed to determine the allowable sampling period and the feasible control gain. A simulation example is given to illustrate the effectiveness of the results.

1. Introduction

During the past decade, distributed coordination of multi-agent systems has received increasing attention. This is largely due to the wide applications of multi-agent systems in engineering, such as networked autonomous vehicles, automated highway systems, formation control, and distributed sensor networks. As an important example of distributed control, there has been significant progress in the study of the consensus problem. Many methods have been developed to solve the consensus problem including algebra graph theory [1–4], linear system theory [5,6], and convex optimization method [7]. In particular, switching topologies were considered in [1–4] in a deterministic framework.

In practice, a stochastic switching model can be used to describe many dynamical systems such as manufacturing systems, communication systems, fault-tolerant systems, and multi-agent systems subject to abrupt changes. In multi-agent systems, the stochastic switching model can be used to describe the interaction topology among the agents. When the topology is stochastically switching, the distributed coordination problem will become very difficult. Very recently, some results on multi-agent systems with Markovian switching topologies have been given in [8–11]. In [8], the authors considered static stabilization of a decentralized discrete-time single-integrator network with Markovian switching topologies. In [9] the mean square consentability problem was studied for a network of double-integrator agents with Markovian switching topologies. In [10,11], the consensus problem was studied for a network of single-integrator agents with Markovian switching topologies in the case of, respectively, undirected information flows and directed information flows. It should be pointed out that there is no leader in the problems studied in [8–11].

When there is a leader or a reference state in the multi-agent team, the consensus problem becomes a coordinated tracking problem or a leader-following consensus problem. The coordinated tracking problem becomes more challenging when the leader is dynamic and only some agents have access to the leader. In [12], the coordination tracking problems with both a time-varying reference state and a constant reference state were studied, where only a subset of the agents has access to the reference state. In [13], a variable structure approach was employed to study a distributed coordinated tracking problem, where only partial measurements of the states of the leader and the followers are available. In [14], the leader-following consensus problem for a multi-agent system with measurement noises and a

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directed interaction topology was studied, where a neighbor-based control scheme with distributed estimators was developed. The leader-following consensus problem for higher-order multi-agent systems with both fixed and switching topologies was studied in [15]. In [16], a coordinated tracking problem was considered for a multi-agent system with variable undirected topologies. In [17], a PD-like discrete-time algorithm was proposed to address the coordinated tracking problem under a fixed topology. However, to the best of the authors' knowledge, few results on coordinated tracking with Markovian switching topologies are available in the existing literature. In this paper, we will extend the coordinated tracking results in [17] to the case of Markovian switching topologies.

The main purpose of this paper is to present a necessary and sufficient condition for the boundedness of the tracking error system. It is assumed that the leader's state is time varying and only some agents can obtain the leader's state. The results presented are mainly based on algebra graph theory and Markovian jump linear system theory. A linear matrix inequality (LMI) approach will be used to derive the allowable sampling period and the feasible control gain. A preliminary version of the current paper has been presented in [18].

Notation. Let $\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the real number set and the nonnegative integer set. Suppose that $A, B \in \mathbb{R}^{p\times p}$. Let $A \succeq B$ (respectively, $A \succ B$) denote that $A-B$ is symmetric positive semi-definite (respectively, symmetric positive definite). Let $\rho(M)$ denote the spectral radius of the matrix M. Let $\operatorname{diag}(A_1, \ldots, A_n)$ denote the diagonal matrix with diagonal block A_i, $i = 1, \ldots, n$. Given $X(k) \in \mathbb{R}^p$, define $\|X(k)\|_E \triangleq \|E[X(k)X^T(k)]\|_2$, where $E[\cdot]$ is the mathematical expectation. Let $|A|$ denote the determinant of the matrix A. Let $\otimes$ represent the Kronecker product of matrices. Let $\mathbf{1}_n$ denote the $n \times 1$ column vector. Let I_n and $\mathbf{0}_{m\times n}$ denote, respectively, the $n \times n$ identity matrix and $m \times n$ zero matrix.

2. Background and preliminaries

2.1. Graph theory notions

Suppose that there exist n followers, labeled as agents 1 to n, and one leader, labeled as agent $n+1$. Let $\bar{\mathcal{G}} \triangleq (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ be a directed graph of order $n+1$ used to model the interaction topology among the n followers and the leader, where $\bar{\mathcal{V}} \triangleq \{1, \ldots, n+1\}$ and $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$ represent, respectively, the node set and the edge set. An edge $(i, j) \in \bar{\mathcal{E}}$ if agent j can obtain information from agent i. Here, agent i is a neighbor of agent j. A directed path is a sequence of edges in a directed graph in the form of $(i_1, i_2), (i_2, i_3), \ldots$, where $i_k \in \bar{\mathcal{V}}$. The union of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph $\mathcal{G}_1 \bigcup \mathcal{G}_2$ with the vertex set $\mathcal{V}(\mathcal{G}_1) \bigcup \mathcal{V}(\mathcal{G}_2)$ and the edge set $\mathcal{E}(\mathcal{G}_1) \bigcup \mathcal{E}(\mathcal{G}_2)$. Let $\bar{\mathcal{A}} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be the adjacency matrix associated with $\bar{\mathcal{G}}$. Here $a_{ij} > 0$ if agent i can obtain information from agent j and $a_{ij} = 0$ otherwise. We assume that there is no self loop in the graph, which implies that $a_{ii} = 0$. We also assume that the leader does not receive information from the followers, which implies that $a_{(n+1)j} = 0, j = 1, \ldots, n$. Let $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ be a directed graph of order n used to model the interaction topology among the n followers. Note that $\mathcal{G}$ is a subgraph of $\bar{\mathcal{G}}$. Also let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be the adjacency matrix associated with $\mathcal{G}$.

In this paper we assume that the interaction topologies are Markovian switching. Let m be a given positive integer. Let $\theta(k)$ be a homogeneous, finite-state, discrete-time Markov chain that takes values in the set $\mathcal{S} \triangleq \{1, \ldots, m\}$, with a probability transition matrix $\Pi = [\pi_{ij}] \in \mathbb{R}^{m\times m}$. In addition, we suppose that the Markov chain is ergodic throughout this paper. Consider a set of directed graphs $\widehat{\mathcal{G}} \triangleq \{\bar{\mathcal{G}}^1, \ldots, \bar{\mathcal{G}}^m\}$, where $\bar{\mathcal{G}}^i$ is a directed graph of order $n+1$ defined as above. By a discrete-time Markovian stochastic graph we understand a map $\mathbf{G}$ from $\mathcal{S}$ to $\widehat{\mathcal{G}}$ such that $\mathbf{G}[\theta(k)] = \bar{\mathcal{G}}^{\theta[k]}$ for all $k \in \mathbb{N}$. Accordingly, $\mathcal{G}^{\theta[k]}$ is the interaction topology among the n followers that is a subgraph of $\bar{\mathcal{G}}^{\theta[k]}$.

2.2. Distributed discrete-time coordinated tracking algorithms

Suppose the dynamics of the ith follower is given by

$$\dot{\xi}_i(t) = u_i(t), \quad i = 1, \ldots, n, \tag{1}$$

where $\xi_i(t) \in \mathbb{R}$ is the state and $u_i(t) \in \mathbb{R}$ is the control input. With zero-order hold $u_i(t) = u_i(kT)$, $kT \le t < (k+1)T$, where k is the discrete-time index, and T is the sampling period, the discretized dynamics of (1) is

$$\xi_i[k+1] = \xi_i[k] + Tu_i[k], \tag{2}$$

where $\xi_i[k]$ and $u_i[k]$ represent, respectively, the state and the control input of the ith follower at $t = kT$.

Let the time-varying leader's state, also called the reference state, be $\xi_{n+1}[k] \equiv \xi^r[k]$. We consider the discrete-time coordinated tracking algorithm adapted from that proposed in [17] as

$$\begin{aligned} u_i[k] &= \frac{1}{\sum_{j=1}^{n+1} a_{ij}^{\theta[k]}} \sum_{j=1}^{n} a_{ij}^{\theta[k]} \left[\frac{\xi_j[k] - \xi_j[k-1]}{T} - \gamma(\xi_i[k] - \xi_j[k]) \right] \\ &\quad + \frac{a_{i(n+1)}^{\theta[k]}}{\sum_{j=1}^{n+1} a_{ij}^{\theta[k]}} \left[\frac{\xi^r[k] - \xi^r[k-1]}{T} - \gamma(\xi_i[k] - \xi^r[k]) \right] \\ &\quad + \frac{\eta - 1}{T}\xi_i[k], \end{aligned} \tag{3}$$

where $a_{ij}^{\theta[k]}$, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$, is the (i, j)th entry of the adjacency matrix $\bar{\mathcal{A}}^{\theta[k]}$ associated with $\bar{\mathcal{G}}^{\theta[k]}$, and γ and η are positive constants. To ensure that the algorithm (3) is well defined, we assume that $\sum_{j=1}^{n+1} a_{ij}^{\theta[k]} \neq 0$, $i = 1, \ldots, n$. That is, each follower has at least one neighbor.[1] Using (3), (2) can be written as

$$\begin{aligned} \xi_i[k+1] &= \eta\xi_i[k] + \frac{T}{\sum_{j=1}^{n+1} a_{ij}^{\theta[k]}} \sum_{j=1}^{n} a_{ij}^{\theta[k]} \\ &\quad \times \left[\frac{\xi_j[k] - \xi_j[k-1]}{T} - \gamma(\xi_i[k] - \xi_j[k]) \right] \\ &\quad + \frac{Ta_{i(n+1)}^{\theta[k]}}{\sum_{j=1}^{n+1} a_{ij}^{\theta[k]}} \left[\frac{\xi^r[k] - \xi^r[k-1]}{T} - \gamma(\xi_i[k] - \xi^r[k]) \right]. \end{aligned} \tag{4}$$

Define the tracking error for follower i as $z_i[k] \triangleq \xi_i[k] - \xi^r[k]$. Denote $Z[k] \triangleq [z_1[k], \ldots, z_n[k]]^T$ and $\zeta[k+1] = [Z^T[k+1], \eta Z^T[k]]^T$, respectively. It follows that

$$\zeta[k+1] = C^{\theta[k]}\zeta[k] + WX^r[k], \tag{5}$$

[1] Due to the tracking nature of the problem (That is, the leaders state is always time varying and only a subset of the followers has access to the leader), it is in general impossible for a group of followers to track the leader under the assumption that some followers might break off from the network even in the deterministic case.

where

$$C^{\theta[k]} \triangleq \begin{bmatrix} (\eta - T\gamma)I_n + (1+T\gamma)D^{\theta[k]}\mathcal{A}^{\theta[k]} & -D^{\theta[k]}\mathcal{A}^{\theta[k]} \\ \eta I_n & \mathbf{0}_{n\times n} \end{bmatrix},$$

$$D^{\theta[k]} \triangleq \mathrm{diag}\left(\frac{1}{\sum_{j=1}^{n+1} a_{1j}^{\theta[k]}}, \ldots, \frac{1}{\sum_{j=1}^{n+1} a_{nj}^{\theta[k]}}\right),$$

$$W \triangleq \begin{bmatrix} I_n \\ \mathbf{0}_{n\times n} \end{bmatrix},$$

$$X^r[k] \triangleq \mathbf{1}_n(\xi_r[k] + \eta\xi_r[k] - \xi_r[k+1] - \xi_r[k-1]),$$

and $\mathcal{A}^{\theta[k]}$ is the adjacency matrix associated with $\mathcal{G}^{\theta[k]}$. According to [19], we know that $\{\zeta[k], k \in \mathbb{N}\}$ is not a Markov process, but the joint process $\{\zeta[k], \theta(k)\}$ is. Here we assume that the reference trajectory is a deterministic signal, and not a random process. The initial state of the joint process is denoted by $\{\zeta_0, \theta_0\}$.

Remark 2.1. Because node j may be not the neighbor of node i at $(k-1)T$, we assume that the each node has the memory function. It can store its state information at $(k-1)T$, and it will send its state information at $(k-1)T$ and kT to his neighbor at kT.

Remark 2.2. In contrast to [17], where the interaction topology is fixed, the interaction topology considered in this paper is Markovian switching. In this case, the coordinated tracking problem becomes more complicated.

3. Convergence analysis

In this section, we analyze (5). When the interaction topology is Markovian switching, the problem becomes very difficult to deal with. We consider a special case, where the interaction topology switches to each graph in $\widehat{\mathcal{G}}$ with an equal probability. In this case the transition probability matrix is $\Pi = \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^T$. In addition, we assume that $0 < \eta < 1$. Denote by $\bar{\mathcal{G}}^u$ (respectively, $\mathcal{G}^u$) the union of $\bar{\mathcal{G}}^1, \ldots, \bar{\mathcal{G}}^m$ (respectively, $\mathcal{G}^1, \ldots, \mathcal{G}^m$). Let $\bar{\mathcal{A}}^u = [a_{ij}^u] \in \mathbb{R}^{(n+1)\times(n+1)}$ (respectively, $\mathcal{A}^u = [a_{ij}^u] \in \mathbb{R}^{n\times n}$) be the adjacency matrix associated with $\bar{\mathcal{G}}^u$ (respectively, $\mathcal{G}^u$). Define $D^u \triangleq \mathrm{diag}(\frac{1}{\sum_{j=1}^{n+1} a_{1j}^u}, \ldots, \frac{1}{\sum_{j=1}^{n+1} a_{nj}^u})$. Before presenting our main result, we need the following lemmas.

Lemma 3.1 ([17, Lemma 3.1]). *Suppose that the leader has directed paths to all followers 1 to n in $\bar{\mathcal{G}}^u$. Then $D^u\mathcal{A}^u$ has all eigenvalues within the unit circle.*

Lemma 3.2. *Suppose that the leader has directed paths to all followers 1 to n in $\bar{\mathcal{G}}^u$. Then $\frac{1}{m}\sum_{i=1}^m D^i\mathcal{A}^i$ has all eigenvalues within the unit circle.*

Proof. Denote $\frac{1}{m}\sum_{i=1}^m D^i\mathcal{A}^i = [\bar{d}_{jl}]$ and $D^u\mathcal{A}^u = [d_{jl}]$. By comparing $\frac{1}{m}\sum_{i=1}^m D^i\mathcal{A}^i$ with $D^u\mathcal{A}^u$, it is easy to see that (1) if $d_{jl} = 1$, then $0 < \bar{d}_{jl} \le 1$; (2) if $d_{jl} < 1$, then $\bar{d}_{jl} < 1$; (3) if $d_{jl} = 0$, then $\bar{d}_{jl} = 0$; (4) if $\sum_{l=1}^n d_{jl} < 1$, then $\sum_{l=1}^n \bar{d}_{jl} < 1$. Hence, by the same method as the proof of Lemma 3.1 in [17], it follows that $\frac{1}{m}\sum_{i=1}^m D^i\mathcal{A}^i$ has all eigenvalues within the unit circle. □

Lemma 3.3 ([19, Proposition 3.6]). *Let $S \triangleq (\Pi^T \otimes I_{4n^2})\mathrm{diag}(C^1 \otimes C^1, \ldots, C^m \otimes C^m)$ and $\bar{S} \triangleq (\Pi^T \otimes I_{2n})\mathrm{diag}(C^1, \ldots, C^m)$, where C^i is defined in (5). If $\rho(S) < 1$, then $\rho(\bar{S}) < 1$.*

Lemma 3.4. *Assume that* $\max(\frac{\sup_k |\xi^r[k]-\xi^r[k-1]|}{T}, \frac{\sup_k |\xi^r[k]-\eta\xi^r[k-1]|}{T}) \le \bar{\xi}$. *Then $\zeta[k]$ is mean-square bounded, that is, $\|\zeta[k]\|_E < \infty$, for all initial ζ_0 and θ_0 if and only if $\rho(S) < 1$, where S is defined in Lemma 3.3.*

Proof. Because $\max(\sup_k |\xi^r[k]-\xi^r[k-1]|, \sup_k |\xi^r[k]-\eta\xi^r[k-1]|) \le T\bar{\xi}$, it follows from (5) that $\|X^r[k]\|$ is bounded. This lemma then directly follows from Theorem 3.34 in [19] and is hence omitted here. □

Lemma 3.5. *Let S be defined in Lemma 3.3. Suppose that $0 < \eta < 1$. For small enough $T\gamma$, $\rho(S) < 1$ if and only if the leader has directed paths to all followers 1 to n in $\bar{\mathcal{G}}^u$.*

Proof (*Sufficiency*). If the leader has directed paths to all followers in $\bar{\mathcal{G}}^u$, it follows from Lemma 3.2 that $\frac{1}{m}\sum_{i=1}^m D^i\mathcal{A}^i$ has all eigenvalues within the unit circle. We will use perturbation arguments to show that $\rho(S) < 1$. Note that C^i in (5) can be written as

$$C^i = M_1^i + T\gamma M_2^i, \tag{6}$$

where

$$M_1^i \triangleq \begin{bmatrix} \eta I_n + D^i\mathcal{A}^i & -D^i\mathcal{A}^i \\ \eta I_n & \mathbf{0}_{n\times n} \end{bmatrix},$$

$$M_2^i \triangleq \begin{bmatrix} D^i\mathcal{A}^i - I_n & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}.$$

Hence S can be written as

$$\begin{aligned} S &= (\Pi^T \otimes I_{4n^2})\mathrm{diag}(C^1 \otimes C^1, \ldots, C^m \otimes C^m) \\ &= (\Pi^T \otimes I_{4n^2})\mathrm{diag}[(M_1^1 + T\gamma M_2^1) \\ &\quad \otimes(M_1^1 + T\gamma M_2^1), \ldots, (M_1^m + T\gamma M_2^m) \\ &\quad \otimes(M_1^m + T\gamma M_2^m)] \\ &= Q_1 + T\gamma Q_2 + T\gamma Q_3 + (T\gamma)^2 Q_4, \end{aligned} \tag{7}$$

where

$$Q_1 \triangleq (\Pi^T \otimes I_{4n^2})\mathrm{diag}(M_1^1 \otimes M_1^1, \ldots, M_1^m \otimes M_1^m),$$
$$Q_2 \triangleq (\Pi^T \otimes I_{4n^2})\mathrm{diag}(M_1^1 \otimes M_2^1, \ldots, M_1^m \otimes M_2^m),$$
$$Q_3 \triangleq (\Pi^T \otimes I_{4n^2})\mathrm{diag}(M_2^1 \otimes M_1^1, \ldots, M_2^m \otimes M_1^m),$$
$$Q_4 \triangleq (\Pi^T \otimes I_{4n^2})\mathrm{diag}(M_2^1 \otimes M_2^1, \ldots, M_2^m \otimes M_2^m).$$

Note that in (7) the last three terms can be treated as small perturbations to the first term when $T\gamma$ is small enough.

Now, we estimate the eigenvalues of Q_1 by elementary transformation. Because $\Pi = \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^T$, by simple calculation, we get that

$$Q_1 = \frac{1}{m}\begin{bmatrix} M_1^1 \otimes M_1^1 & M_1^2 \otimes M_1^2 & \cdots & M_1^m \otimes M_1^m \\ \vdots & \vdots & \vdots & \vdots \\ M_1^1 \otimes M_1^1 & M_1^2 \otimes M_1^2 & \cdots & M_1^m \otimes M_1^m \end{bmatrix}. \tag{8}$$

Denote the elementary transformation block matrices $\mathcal{P}_1 \in \mathbb{R}^{4mn^2\times 4mn^2}$ and $\mathcal{P}_2 \in \mathbb{R}^{4n^2\times 4n^2}$ as, respectively,

$$\mathcal{P}_1 \triangleq \begin{bmatrix} I_{4n^2} & \mathbf{0}_{2n\times 2n} & \cdots & I_{4n^2} \\ \mathbf{0}_{2n\times 2n} & I_{4n^2} & \cdots & I_{4n^2} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{2n\times 2n} & \mathbf{0}_{2n\times 2n} & \cdots & I_{4n^2} \end{bmatrix},$$

$$\mathcal{P}_2 \triangleq \begin{bmatrix} I_{2n^2} & \mathbf{0}_{2n^2\times 2n^2} \\ I_{2n^2} & I_{2n^2} \end{bmatrix}.$$

Then the equation in Box I follows.

To study the roots of $|\lambda I_{2n} - \frac{1}{m}\eta\sum_{i=1}^m M_1^i|$, note that

$$\left|\lambda I_{2n} - \frac{1}{m}\eta\sum_{i=1}^m M_1^i\right|$$

$$
\begin{aligned}
|\lambda I_{4mn^2} - Q_1| &= |\mathcal{P}_1^{-1}(\lambda I_{4mn^2} - Q_1)\mathcal{P}_1| \\
&= \lambda^{4(m-1)n^2} \left| \lambda I_{4n^2} - \frac{1}{m}\sum_{i=1}^{m}(M_1^i \otimes M_1^i) \right| \\
&= \lambda^{4(m-1)n^2} \begin{vmatrix} \Omega_1 & \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i) \\ -\dfrac{1}{m}\eta I_n \otimes \displaystyle\sum_{i=1}^{m} M_1^i & \lambda I_{2n^2} \end{vmatrix} \\
&= \lambda^{4(m-1)n^2} \left| \mathcal{P}_2^{-1} \begin{bmatrix} \Omega_1 & \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i) \\ -\dfrac{1}{m}\eta I_n \otimes \displaystyle\sum_{i=1}^{m} M_1^i & \lambda I_{2n^2} \end{bmatrix} \mathcal{P}_2 \right| \\
&= \lambda^{4(m-1)n^2} \begin{vmatrix} \lambda I_{2n^2} - \dfrac{1}{m}\eta I_n \otimes \displaystyle\sum_{i=1}^{m} M_1^i & \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i) \\ \mathbf{0}_{2n^2\times 2n^2} & \lambda I_{2n^2} - \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i) \end{vmatrix} \\
&= \lambda^{4(m-1)n^2} \left| \lambda I_{2n^2} - \frac{1}{m}\eta I_n \otimes \sum_{i=1}^{m} M_1^i \right| \left| \lambda I_{2n^2} - \frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i) \right| \\
&= \lambda^{4(m-1)n^2} \left| \lambda I_{2n} - \frac{1}{m}\eta \sum_{i=1}^{m} M_1^i \right|^n \left| \lambda I_{2n^2} - \frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i) \right|, \qquad (9)
\end{aligned}
$$

where

$$\Omega_1 \triangleq \lambda I_{2n^2} - \frac{1}{m}\eta I_n \otimes \sum_{i=1}^{m} M_1^i - \frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i).$$

Box I.

$$
\begin{aligned}
&= \begin{vmatrix} (\lambda-\eta)I_n - \dfrac{1}{m}\eta\displaystyle\sum_{i=1}^{m} D^i\mathcal{A}^i & \dfrac{1}{m}\eta\displaystyle\sum_{i=1}^{m} D^i\mathcal{A}^i \\ -\eta I_n & \lambda I_n \end{vmatrix} \\
&= (\lambda-\eta)^n \left| \lambda I_n - \frac{1}{m}\eta\sum_{i=1}^{m} D^i\mathcal{A}^i \right|. \qquad (10)
\end{aligned}
$$

Because $0 < \eta < 1$ and $\rho(\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i) < 1$, all roots of (10) are within the unit circle.

Next we show that the roots of $|\lambda I_{2n^2} - \frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)|$ are within the unit circle (i.e., $\rho[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)] < 1$) by showing that $\lim_{s\to\infty}[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)]^s = \mathbf{0}_{2n^2\times 2n^2}$. Denote $D^i\mathcal{A}^i = [d_{jl}^i]$ and $\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i = [\bar{d}_{jl}]$. We have the equation in Box II.

It is easy to see that $\frac{1}{m}\sum_{i=1}^{m} d_{jl}^i = \bar{d}_{jl} \geq 0, j, l = 1, \ldots, n$. We first let $s = 2$. By computation we find that the (j, l)th block entry of $[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)]^2$ is $\sum_{k=1}^{n}[\frac{1}{m}\sum_{i=1}^{m}(d_{jk}^i M_1^i)][\frac{1}{m}\sum_{i=1}^{m}(d_{kl}^i M_1^i)]$. The sum of the coefficients of $M_1^i M_1^j$, $i, j = 1, \ldots, m$, is equal to $\sum_{k=1}^{n}(\frac{1}{m}\sum_{i=1}^{m} d_{jk}^i)(\frac{1}{m}\sum_{i=1}^{m} d_{kl}^i)$. We can find a matrix $\widehat{M} = \begin{bmatrix} I_n + \widehat{D}\widehat{\mathcal{A}} & -\widehat{D}\widehat{\mathcal{A}} \\ I_n & \mathbf{0}_{n\times n} \end{bmatrix}$ such that the maximum absolute value of all entries of $(\widehat{M})^2$ is greater than or equal to that of $M_1^i M_1^j$, $i, j = 1, \ldots, m$. Here $\widehat{D}\widehat{\mathcal{A}}$ is defined analogously as $D^i\mathcal{A}^i$ and the corresponding graph has the same vertex set as that of $D^i\mathcal{A}^i$. On the other hand, we know that the coefficient of the (j, l)th block entry $(\widehat{M})^2$ of $[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes \widehat{M})]^2 = [(\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i) \otimes \widehat{M}]^2$ is also $\sum_{k=1}^{n}(\frac{1}{m}\sum_{i=1}^{m} d_{jk}^i)(\frac{1}{m}\sum_{i=1}^{m} d_{kl}^i)$. We thus have that the maximum absolute value of all entries of $[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)]^2$ is less than or equal to that of $[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes \widehat{M})]^2$. Using the same method, we can find an $\widehat{M}$ such that the same conclusion holds for $s > 2$. By simple calculation we get that $\rho(\widehat{M}) \leq 1$. In addition, note from Lemma 3.2 that $\rho(\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i) < 1$. It follows from the property of the Kronecker product that $\rho[(\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i) \otimes \widehat{M}] < 1$. Hence, $\lim_{s\to\infty}[\frac{1}{m}(\sum_{i=1}^{m} D^i\mathcal{A}^i \otimes \widehat{M})]^s = \lim_{s\to\infty}[(\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i) \otimes \widehat{M}]^s = \mathbf{0}_{2n^2\times 2n^2}$. Therefore, we conclude that $\lim_{s\to\infty}[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)]^s = \mathbf{0}_{2n^2\times 2n^2}$, which implies that $\rho[\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i \otimes M_1^i)] < 1$.

From the above discussion, we know that all eigenvalues of Q_1 are within the unit circle. For small enough $T\gamma$, the last three perturbation terms in (7) can be neglected. Hence it follows that for small enough $T\gamma$, $\rho(S) < 1$.

(Necessity). For necessity, we need to prove that $\rho(S) \geq 1$ for any $T > 0$ and $\gamma > 0$ if the leader has no directed paths to all followers. From Lemma 3.3, we only need to prove that $\rho(\bar{S}) \geq 1$ for any $T > 0$ and $\gamma > 0$, where $\bar{S}$ is defined in Lemma 3.3. If the leader has no directed paths to some followers in $\bar{\mathcal{G}}^u$, then these followers receive information from neither the leader nor the other followers in each $\bar{\mathcal{G}}^i, i = 1, \ldots, m$. We assume that there are l such followers. Each of these l followers must have at least one neighbor due to the assumption mentioned after (3). Without loss of generality, we assume that followers 1 to l are such l followers. In this case, $\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i$ has the following form:

$$\begin{bmatrix} A_{11} & \mathbf{0}_{l\times(n-l)} \\ A_{21} & A_{22} \end{bmatrix}. \qquad (12)$$

Therefore, the eigenvalues of $\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i$ are those of A_{11} together with those of A_{22}. According to the definition of $\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i$, we know that A_{11} is a row stochastic matrix. Hence 1 is an eigenvalue of A_{11} with an associated right eigenvector $\mathbf{1}_l$. Let μ_i be the ith eigenvalue of $\frac{1}{m}\sum_{i=1}^{m} D^i\mathcal{A}^i$. Without loss of generality, let $\mu_1 = 1$.

$$\frac{1}{m}\sum_{i=1}^{m}(D^i\mathcal{A}^i\otimes M_1^i)=\begin{bmatrix}\mathbf{0}_{2n\times 2n} & \frac{1}{m}\sum_{i=1}^{m}(d_{12}^iM_1^i) & \cdots & \frac{1}{m}\sum_{i=1}^{m}(d_{1n}^iM_1^i)\\ \vdots & \vdots & \vdots & \vdots\\ \frac{1}{m}\sum_{i=1}^{m}(d_{n1}^iM_1^i) & \frac{1}{m}\sum_{i=1}^{m}(d_{n2}^iM_1^i) & \cdots & \mathbf{0}_{2n\times 2n}\end{bmatrix}. \tag{11}$$

Box II.

Next we consider the eigenvalues of $\bar{S}$. Denote the elementary block matrix $\bar{\mathcal{P}}\in\mathbb{R}^{2mn\times 2mn}$ as

$$\bar{\mathcal{P}}\triangleq\begin{bmatrix}I_{2n} & \mathbf{0}_{2n\times 2n} & \cdots & I_{2n}\\ \mathbf{0}_{2n\times 2n} & I_{2n} & \cdots & I_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \mathbf{0}_{2n\times 2n} & \mathbf{0}_{2n\times 2n} & \cdots & I_{2n}\end{bmatrix}.$$

Then, it can be computed that

$$\begin{aligned}&|\lambda I_{2mn}-\bar{S}|\\ &=|\lambda I_{2mn}-\bar{\mathcal{P}}^{-1}\bar{S}\bar{\mathcal{P}}|\\ &=\lambda^{2(m-1)n}\left|\lambda I_{2n}-\frac{1}{m}\sum_{i=1}^{m}C^i\right|\\ &=\lambda^{2(m-1)n}\begin{vmatrix}\Omega & \frac{1}{m}\sum_{i=1}^{m}D^i\mathcal{A}^i\\ -\eta I_n & \lambda I_n\end{vmatrix}\\ &=\lambda^{2(m-1)n}\prod_{i=1}^{n}\{\lambda^2+[T\gamma-\eta-(1+T\gamma)\mu_i]\lambda+\eta\mu_i\},\end{aligned}$$

where

$$\Omega\triangleq\lambda I_n-(\eta-T\gamma)I_n-\frac{1+T\gamma}{m}\sum_{i=1}^{m}D^i\mathcal{A}^i.$$

By some simple computation we have that $\lambda_{1,2}=1,\eta$, when $\mu_1=1$. It then follows from the above computation that $\rho(\bar{S})\geq 1$ for any $T>0$ and $\gamma>0$. $\square$

Remark 3.1. Lemma 3.5 provides a necessary and sufficient condition for $\rho(S)<1$ under the assumption that $0<\eta<1$. It is worth pointing out that $0<\eta<1$ is not necessary in the proof of necessity.

Based on the above discussion, we now summarize the main result in the following theorem.

Theorem 3.2. *Suppose that the reference state $\xi^r[k]$ satisfies that $\max\left(\frac{\sup_k|\xi^r[k]-\xi^r[k-1]|}{T},\frac{\sup_k|\xi^r[k]-\eta\xi^r[k-1]|}{T}\right)\leq\bar{\xi}$ and $0<\eta<1$.*[2] *Then for small enough $T\gamma$, the tracking errors for the n followers are ultimately mean-square bounded if and only if the leader has directed paths to all followers 1 to n in $\bar{\mathcal{G}}^u$. In particularly, there exist $0<\alpha<1$ and $\beta\geq 1$ such that the ultimate bound for $\|\zeta[k]\|_E$ is given by $2nT\bar{\xi}\frac{\beta}{1-\alpha}$.*

Proof. It follows from (5) that

$$\zeta[k]=C^{\theta[k-1]}\cdots C^{\theta[0]}\zeta_0+WX^r[k-1]+\sum_{l=0}^{k-2}C^{\theta[k-1]}\cdots C^{\theta[l+1]}WX^r[l]. \tag{13}$$

Then we have that

$$\|\zeta[k]\|_E\leq\|C^{\theta[k-1]}\cdots C^{\theta[0]}\zeta_0\|_E+\|WX^r[k-1]\|_E+\sum_{l=0}^{k-2}\|C^{\theta[k-1]}\cdots C^{\theta[l+1]}WX^r[l]\|_E. \tag{14}$$

Note that $WX^r(l)$ is deterministic and $|\xi^r(k)+\eta\xi^r(k)-\xi^r(k+1)-\xi^r(k-1)|\leq 2T\bar{\xi}$. We thus obtain that

$$\|WX^r(k-1)\|_E\leq 2\sqrt{n}T\bar{\xi}. \tag{15}$$

Based on Lemmas 3.4 and 3.5, and according to Theorem 3.9 in [19], we know that there exist $0<\alpha_1<1$ and $\beta_1\geq 1$ such that

$$\|C^{\theta[k-1]}\cdots C^{\theta[0]}\zeta_0\|_E\leq\sqrt{2n\alpha_1^k\beta_1}\|\zeta_0\|_2, \tag{16}$$

$$\|C^{\theta[k-1]}\cdots C^{\theta[l+1]}WX^r[l]\|_E\leq 2nT\bar{\xi}\sqrt{2\alpha_1^{k-l-1}\beta_1}. \tag{17}$$

Denote $\alpha\triangleq\sqrt{\alpha_1}$ and $\beta\triangleq\sqrt{2\beta_1}$. Note that $2\sqrt{n}T\bar{\xi}\leq 2nT\bar{\xi}\beta$. It thus follows from (14) to (17) that

$$\|\zeta[k]\|_E\leq\sqrt{n}\alpha^k\beta\|\zeta_0\|_2+2nT\bar{\xi}\frac{\beta(1-\alpha^k)}{1-\alpha}.$$

Therefore, the ultimate mean-square bound is given by $2nT\bar{\xi}\frac{\beta}{1-\alpha}$. $\square$

Remark 3.3. Theorem 3.2 provides a necessary and sufficient condition for the boundedness of the tracking error system (5). In the theorem we require $T\gamma$ to be small enough. Next we provide a method to compute the allowable $T\gamma$. It follows from Theorem 3.9 in [19] that $\rho(S)<1$ is equivalent to that there exist symmetric positive-definite matrices $P_i\in\mathbb{R}^{2n\times 2n}$ such that

$$P_i-(C^i)^T\left(\frac{1}{m}\sum_{j=1}^{m}P_j\right)C^i\succ\mathbf{0}_{2n\times 2n},\quad i=1,\ldots,m. \tag{18}$$

Then by applying Schur complement lemma, it follows that (18) is equivalent to

$$\begin{bmatrix}P_i & (C^i)^T\\ C^i & \left(\frac{1}{m}\sum_{j=1}^{m}P_j\right)^{-1}\end{bmatrix}\succ\mathbf{0}_{4n\times 4n},\quad i=1,\ldots,m. \tag{19}$$

Note that (19) is not a linear matrix inequality (LMI) because of the term $(\frac{1}{m}\sum_{j=1}^{m}P_j)^{-1}$. Denote $Q_i=(\frac{1}{m}\sum_{j=1}^{m}P_j)^{-1}$. Then we can convert the non-convex problem (19) to a minimization problem with LMI constraints, namely,

$$\min\operatorname{tr}\left[\sum_{i=1}^{m}\left(\frac{1}{m}\sum_{j=1}^{m}P_j\right)Q_i\right]$$

subject to

$$\begin{bmatrix}P_i & (C^i)^T\\ C^i & Q_i\end{bmatrix}\succ\mathbf{0}_{4n\times 4n},\qquad\begin{bmatrix}\frac{1}{m}\sum_{j=1}^{m}P_j & I_n\\ I_n & Q_i\end{bmatrix}\succeq\mathbf{0}_{4n\times 4n},$$

$$P_i\succ\mathbf{0}_{2n\times 2n},\qquad Q_i\succ\mathbf{0}_{2n\times 2n}.$$

[2] Under these assumptions, we can obtain that $\xi^r[k]$ is bounded.

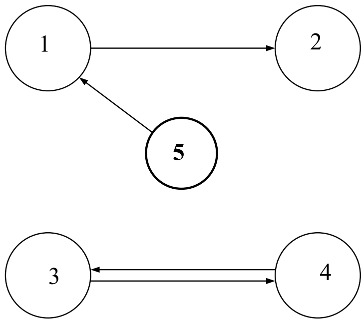

Fig. 1. Directed topology $\mathcal{G}_1$.

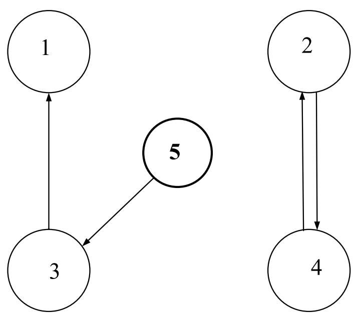

Fig. 2. Directed topology $\mathcal{G}_2$.

If the solution to the above minimization problem is $2mn$, then we can get the allowable $T\gamma$. The proposed minimization problem can be solved by the cone complementary linearization (CCL) method in [20], which can also be found in the literature such as [9,21].

Remark 3.4. Note that $0 < \eta < 1$ is not necessary in the proof of necessity. Therefore, it is possible that η takes a value greater than or equal to 1. When we apply the method in Remark 3.3, we can let $0 < \eta < 1$ or $\eta \geq 1$. If there is a solution to the minimization problem in Remark 3.3, the given η is allowable.

Remark 3.5. Note that the current paper focuses on solving a distributed tracking problem with a *dynamic* leader under the constraint that the leader is a neighbor of only a subset of the followers while [11] focuses on a stationary leaderless consensus problem, where the final consensus value is a constant. Note that the leader's state is changing over time but its state or state derivative is not available to all followers. As a result, a more stringent connectivity condition is required in this paper. Actually, if the graph is only jointly connected as in [11], in general it is impossible to ensure distributed tracking under the constraint of the current paper. Of course, if the leader is stationary (that is, the state of the leader is constant), then the tracking problem here can be considered a special case of the stationary consensus problem under a directed network topology as studied in [11]. In this case, a standard proportional algorithm (instead of a proportional and derivative algorithm studied in this paper) and the joint connectivity requirement are sufficient.

4. Simulations

In this section, we provide an example to demonstrate the effectiveness of the proposed algorithm. For simplicity, we let $a_{ij}^{\theta[k]} = 1$ if $(j, i) \in \bar{\mathcal{E}}^{\theta[k]}$, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$. We assume that there exist one leader and four followers. We also assume that the Markov chain has two modes with the corresponding graphs shown in Figs. 1 and 2, respectively. It can be seen that the leader has no directed path to all followers in each topology. However, the leader has directed paths to all followers in the union topology of $\bar{\mathcal{G}}^1$ and $\bar{\mathcal{G}}^2$. We let the transition probability matrix be $\Pi = \frac{1}{2}\mathbf{1}_2\mathbf{1}_2^T$.

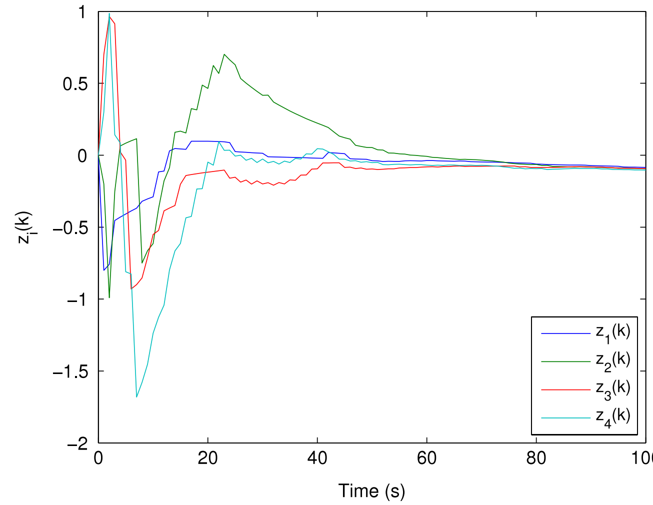

Fig. 3. Tracking errors using (3) ($T = 0.001$, $\gamma = 1.7$).

First, let $\eta = 0.95$. By applying the CCL algorithm and the Matlab LMI toolbox we obtain that a feasible $T\gamma$ is $T\gamma = 0.0017$. In this case, $\rho(S) = 0.8993 < 1$. It follows that $\rho(S) < 1$ for all $T\gamma \leq 0.0017$. The time-varying reference state is chosen as $\xi^r[k] = \sin[kT]$. Let $T = 0.001$ and $\gamma = 1.7$ ($T\gamma = 0.0017$). The tracking errors are shown in Fig. 3. It can be seen that the ultimate tracking errors are very small. Second, let $\eta = 1$. We obtain that a feasible $T\gamma$ is $T\gamma = 0.0304$. In this case, $\rho(S) = 0.9409 < 1$, which implies that $0 < \eta < 1$ is not a necessary assumption. It follows that $\rho(S) < 1$ for all $T\gamma \leq 0.0304$. The time-varying reference state is chosen as $\xi^r[k] = \sin[kT] + kT$. We first let $T = 0.01$ and $\gamma = 3.04$ ($T\gamma = 0.0304$). The tracking errors are shown in Fig. 4. It can be seen that the ultimate tracking errors are very small. Then we let $T = 0.1$ and $\gamma = 0.304$ ($T\gamma = 0.0304$). The tracking errors are shown in Fig. 5. It can be seen that the ultimate tracking errors are larger, which shows that the tracking errors are related to the sampling period T.

5. Conclusion and future work

In this paper, we have studied the distributed discrete-time coordinated tracking problem for multi-agent systems with Markovian switching topologies. The time-varying reference state has been considered. Based on algebraic graph theory and Markovian jump linear system theory, the necessary and sufficient conditions for the boundedness of the tracking errors have been obtained. An LMI approach has been used to find proper sampling periods and control gains. We have assumed that the topology switching probabilities are equal. The general case where the switching probabilities are not necessarily equal will be addressed in our future work.

Acknowledgments

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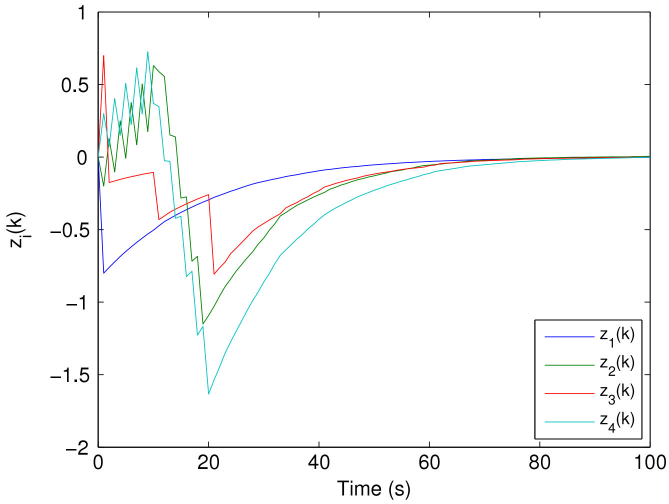

Fig. 4. Tracking errors using (3) ($T = 0.01$, $\gamma = 3.04$).

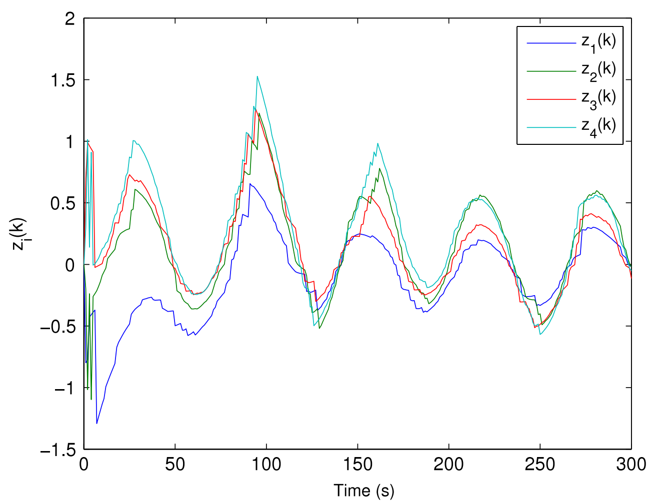

Fig. 5. Tracking errors using (3) ($T = 0.1$, $\gamma = 0.304$).

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