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Series solutions for magnetohydrodynamic flow of non-Newtonian nanofluid and heat transfer in coaxial porous cylinder with slip conditions

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Abstract

A study on the flow of non-Newtonian nanofluid between two coaxial cylinders is made. Two types of series solutions are constructed by choosing constant and variable viscosity. The effects of heat transfer analysis on nanoparticles in the presence of magnetohydrodynamics, porosity and partial slip are also examined. To drive the solutions of nonlinear boundary value problems, we have used a recently developed method, the optimal homotopic asymptotic method, which has been proved an effective technique for solving nonlinear equations. Comparison with existing, documented results through reduction of emerging parameters reveals that the presented series solutions are correct. The solution valid for the no-slip condition for all values of the non-Newtonian parameters can be derived as special case of the present analysis. Finally, the influence of pertinent parameters on velocity, temperature and nanoparticle concentration is discussed and illustrated in graphical form.

Keywords

Non-Newtonian nanofluid, heat transfer, porous media, magnetohydrodynamics, partial slip, nonlinear coupled equations, series solutions

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Introduction

During the past few decades, the flows of non-Newtonian fluids[1–4] have been recognized as more promising in industry and technology; for instance in the processing of polymers, biomechanics, enhanced oil recovery, food products, etc. But very little efforts are devoted to examining non-Newtonian nanofluids. Since nanotechnology has been widely used in industry, nanofluids are therefore a topic of great interest to researchers nowadays. Nanofluids are fluids with added nanoscale particles, a term which was first introduced by Choi.[5] Choi et al.[6] showed that the addition of a small amount, less than 1% by volume, of nanoparticles to a convectional heat transfer liquid increased the thermal conductivity of the fluid up to approximately two times. Khanafer et al.[7] seem to have been the first to examine the heat transfer performance of nanofluids inside an enclosure taking into account the solid particle dispersion. After these studies, nanotechnology is considered by many to be one of the significant forces that will drive the next major industrial revolution of this century. It represents the most relevant technological cutting edge currently being explored. It aims at manipulating the structure of matter at the molecular level with the goal for innovation in virtually every industry and public endeavor,

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including biological sciences, physical sciences, electronics cooling, transportation, the environment and national security. Some experimental and numerical studies on nanofluids can be found in Lotfi et al.[8] and Ahmad and Pop[9] and several references therein.

Moreover, the classical Navier–Stokes equations have been proved to be inadequate to describe and capture the characteristics of complex rheological fluids. Mostly, the equations that govern the flow of non-Newtonian nanofluids are of higher order, much more complicated and subtle in comparison with Navier–Stokes equations. Hence, the mechanics of non-Newtonian nanofluids presents special challenges to mathematicians, engineers and numerical analysts since the governing equations of non-Newtonian nanofluids are highly nonlinear and cannot be solved by using conventional methods. To accomplish this task and solve such problems, perturbation is an important tool. Nayfeh and Mook[10] have presented an account of various perturbation techniques, pointing out their similarities, differences, advantages and limitations as well. It is well known that mostly perturbation techniques are based on small parameters in governing equations or boundary conditions, called perturbation quantities. The small parameter plays a very important role because it determines not only the accuracy of the perturbation approximations but also the validity of the perturbation method itself. In general, it is not guaranteed that a perturbation result is valid in the whole region for all physical parameters. Therefore, it is necessary to develop some new techniques that are independent of small parameters because in the physical sciences very many nonlinear problems exist which do not contain any small parameter, in particular those having strong nonlinearity. To overcome the restrictions of perturbation techniques, some powerful mathematical methods have been successfully introduced to eliminate the small parameter, such as the artificial parameter method introduced by He,[11] the tanh method,[12] the Jacobi elliptic function method,[13] Adomian's decomposition method,[14] the homotopy perturbation method,[15] the modified homotopy perturbation method,[16] the variational iteration method,[17] the iteration perturbation method,[18] and so on. In principle, all of these methods are based on a so-called artificial parameter in which approximate solutions are expanded into series of such kind of artificial parameter. This artificial parameter is often used in such a way that one can easily get approximation solutions efficiently for a given nonlinear equation. All of these traditional methods cannot provide any guarantee for the convergence of approximation series. In 1992 Liao[19] took the lead to apply homotopy,[20] a basic concept in topology,[21] to get analytic approximations of nonlinear equations and introduced a non-auxiliary parameter[22] to control the convergence region. Recently Marinca et al.[23] developed a very interesting method, the optimal homotopic asymptotic method (OHAM), to approximate the solution of nonlinear problems in the frame of the homotopy analysis method. This method is not only valid for small (or large) values of physical parameter but also minimizes the residual error, which shows its validity and great potential to solve the nonlinear problems in science and engineering. The application of this method in fluid mechanics, heat and mass transfer analysis has been successfully studied by Marinca et al.[24–25]

Furthermore, the porous medium is used effectively in many industrial applications[26] such as heat pipes, solid matrix heat exchangers, electronic cooling and chemical reactors. For a solar collector with air or water as the working fluid, a porous medium can provide an effective means for thermal energy storage. During the period of charging and recovery, transient thermal response aspects of the process for the packed bed are of major concern. It is also well accepted now that slip effects may appear for two types of fluids (i.e. rare field gases[27] and fluids having much more elastic character). In these fluids, slippage appears subject to large tangential traction. It is noticed through experimental observations[28–32] that the occurrence of slippage is possible in non-Newtonian fluids, such as polymer solution and molten polymer. In addition, a clear layer is sometimes found next to the wall when flow of a dilute suspension of particles are examined. In experimental physiology such a layer is observed when blood flow through capillary vessels is studied.[33] The fluids that exhibit slip effects have many applications, for instance the polishing of artificial heart valves and internal cavities,[34] and are supported by the molecular theories[35,36] as well.

The purpose of the present paper is to revisit worthwhile problems[37–39] in order to analyze the effects of magnetohydrodynamics (MHD) and slip boundary conditions in coaxial porous cylinders for non-Newtonian nanofluids. We offer the series solutions of velocity and temperature fields by making choices of constant and variable viscosity. It is well known that the said nonlinear problems are usually solved by numerical methods but here we intend to solve this problem using the relatively new method of OHAM. Comparison with the previous relevant studies is also made.

Modeling of OHAM

In this section we look at the basic idea of the OHAM. We start by classifying the equations to be solved for velocity v, temperature θ and nanoparticle concentration ϕ, that is

$$\begin{aligned} &L_1(v(r)) + g_1(r) + N_1(v(r), \theta(r), \phi(r)) = 0 \\ &L_2(\theta(r)) + g_2(r) + N_2(u(r), \theta(r), \phi(r)) = 0 \\ &L_3(\phi(r)) + g_3(r) + N_2(u(r), \theta(r), \phi(r)) = 0 \\ &B(u, \theta, \phi) = 0 \end{aligned} \tag{1}$$

where $L_i(i = 1 - 3)$ is a nonlinear operator, $g_i(i = 1 - 3)$ is a known function, $N_i(i = 1 - 3)$ is a nonlinear operator and B is a boundary operator.

By means of OHAM we first construct a homotopy $\varphi(r, p) : R\times[0, 1] \to R$ which satisfies

$$(1 - p)\,[L_1(\varphi_1(r, p)) + g_1(r)] = H(p)[L_1(\varphi_1(r, p)) + G_{r1}(r) + N_1(\varphi_1(r, p))]B_1(\varphi_1(r, p)) = 0 \quad (2)$$

$$(1 - p)\,[L_2(\varphi_2(r, p)) + g_2(r)] = H(p)[L_2(\varphi_2(r, p)) + G_{r2}(r) + N_2(\varphi_2(r, p))]B_2(\varphi_2(r, p)) = 0 \quad (3)$$

$$(1 - p)\,[L_3(\varphi_3(r, p)) + g_3(r)] = H(p)[L_3(\varphi_3(r, p)) + G_{r3}(r) + N_3(\varphi_3(r, p))]B_3(\varphi_3(r, p)) = 0 \quad (4)$$

where $r \in R$ and $0 \leqslant p \leqslant 1$ is an embedding parameter, $H(p)$ is a non-zero auxiliary function for $p \neq 0$ and $H(0) = 0$, and $\varphi_i(r, p)(i = 1 - 3)$ are unknown functions. Obviously for $p = 0$ and $p = 1$, we have

$$\begin{aligned} &\varphi_1(r, 0) = v_0(r), \quad \varphi_2(r, 0) = \theta_0(r), \quad \varphi_3(r, 0) = \phi_0(r) \\ &\varphi_1(r, 1) = v(r), \quad \varphi_2(r, 1) = \theta(r), \quad \varphi_3(r, 1) = \phi(r) \end{aligned} \quad (5)$$

Now we choose the auxiliary function $H(p)$ in the form

$$H(p) = pC_1 + p^2C_2 + \ldots \quad (6)$$

where $C_1, C_2, \ldots$ are constants.

By Taylor's theorem one can write equation (2) to get $\varphi_i(r, p, C_j)$ in the following form

$$\varphi_i(r, p, C_j) = u_i(r) + \sum_{k \geqslant 1} u_{ik}(r, C_j)\,p^k, \quad j = 1, 2, \ldots \quad (7)$$

Using equation (7) in equation (2) and equating the like terms of p, we obtain the following linear equations in which u_1, u_2 and u_3 represent v, θ and φ respectively.

Using equations (2) to (4) and equations (6) and (7) we have respectively zeroth order and first order problems as follows

$$L_i(u_{i0}(r)) + g_i(r) = 0, \qquad B(u_{i0}) = 0 \quad (8)$$

$$L_i(u_{i1}(r)) = C_1 N_{i0}(u_{i0}(r)), \qquad B(u_{i1}) = 0 \quad (9)$$

and the kth order equation is defined by

$$\begin{aligned} &L_i(u_k(r) - u_{k-1}(r)) = C_k N_{i0}(u_0(r)) \\ &+ \sum_{j=1}^{k-1} C_j\left[L_i(u_{k-j}(r)) + N_{i(k-j)}(u_0(r), u_1(r), u_{k-j}(r))\right] \\ &B\left(u_k, \frac{\mathrm{d}u_k}{\mathrm{d}r}\right) = 0; \qquad k = 2, 3, 4\ldots \end{aligned} \quad (10)$$

where

$$N_i(u_i(r)) = N_{i0}(u_{i0}(r)) + pN_{i1}(u_{i0}(r), u_{i1}(r)) + p^2 N_{i2}(u_{i0}(r), u_{i1}(r), u_{i2}(r)) + \ldots \quad (11)$$

The solution of equations (2) to (4) can be determined approximately in the form

$$u_i^{(m)}(r, p, C_j) = u_{i0}(r) + \sum_{k=1}^{m} u_{ik}(r, C_j) \quad (12)$$

To find the value of C_j, we substitute equation (12) into equations (2) to (4) and as a result we get the following residual

$$R_i(r, C_j) = L_i\left(u_i^{(m)}(r)\right) + g_i(r) + N_i\left(u_i^{(m)}(r), \theta(r), \phi(r)\right) \quad (13)$$

If $R(r, C_j) = 0$ then $u^{(m)}(r, C_j)$ happens to be the exact solution. Generally such case will not arise for nonlinear problems, but we can minimize the functional by

$$J_i(C_j) = \int_a^b R_i^2(r, C_j)\,\mathrm{d}r \quad (14)$$

where a and b are two values, depending on the given problem, for locating the desired C_j. Finally the unknown constants $C_j(j = 1, 2, 3, \ldots, m)$ can be optimally identified from the conditions

$$\frac{\partial J_i}{\partial C_j} = 0 \quad (15)$$

With these constants known, the approximate solution of order m is now well determined.

Modeling for non-Newtonian nanofluids

We start with the basic fact that the flow of a viscous fluid is governed by continuity and Navier–Stokes equations which, when the fluid is incompressible, take the following three field equations that embody the conservation of total mass, momentum, thermal energy and nanoparticles, respectively

$$\begin{aligned} \rho_f\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V}\right) &= \mathrm{div}\mathbf{T} - \frac{\mu\varphi}{k}\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\mathbf{V} \\ &+ \left[\phi\rho_p + (1 - \phi)\,\rho_f[1 - \beta_T(\theta - \theta_w)]\right] g \end{aligned} \quad (16)$$

$$\begin{aligned} (\rho c)_f\left(\frac{\partial \theta}{\partial t} + \mathbf{V}.\nabla\theta\right) &= k\nabla^2\theta \\ &+ (\rho c)_p\left[D_b\nabla\phi.\nabla\theta + \frac{D_T}{\theta_w}\nabla\theta.\nabla\theta\right] \end{aligned} \quad (17)$$

$$\left(\frac{\partial \phi}{\partial t} + \mathbf{V}.\nabla\phi\right) = D_b\nabla^2\phi + \frac{D_T}{\theta_w}\nabla^2\theta \quad (18)$$

along with the boundary conditions

$$v(R_1) = v_0, \quad v(R_2) = \gamma\left[\frac{dv}{dr}(R_2)\right] 0$$
$$\theta(R_1) = \theta_w, \quad \theta(R_2) = 0$$
$$\phi(R_1) = \phi_w \quad \phi(R_2) = 0 \tag{19}$$

where $\mathbf{V}[0, 0, u(r)]$ is the velocity, θ is the temperature, ϕ is the volume fraction of nanoparticles, ρ_f is the density of the base fluid, ρ_p is the density of the nanoparticles, g is the gravitational acceleration, μ is viscosity, k is thermal conductivity, β_T is the volumetric solutal expansion coefficient of the nanofluid, and R_1 and R_2 are radii of the inner and outer cylinders respectively. The Brownian diffusion coefficient and the thermophoretic diffusion coefficient are respectively denoted by D_b and D_T.

For third grade fluid the Cauchy stress tensor $\mathbf{T}$ is defined by

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1 \tag{20}$$

where p_1 is hydrostatic pressure, μ is the dynamic viscosity, $\mathbf{I}$ is the identity tensor and $\alpha_i(i = 1, 2)$ and $\beta_j(j = 1-3)$ are material constants. The Rivlin–Ericksen tensors are given by

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^t \tag{21}$$

$$\mathbf{A}_n = \frac{d\,\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}\nabla\mathbf{V} + (\nabla\mathbf{V})^t A_{n-1}, \qquad n > 1 \tag{22}$$

in which ∇ is the gradient operator. Moreover, thermodynamics imposes the following constraints

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3}$$
$$\beta_1 = \beta_2 = 0, \quad \beta_3 \geqslant 0 \tag{23}$$

Using equations (16) to (23) and the following non-dimensional quantities

$$\bar{u} = \frac{u}{u_0}, \quad \bar{r} = \frac{r}{R}, \quad \bar{\theta} = \frac{\theta - \theta_w}{\theta_m - \theta_0},$$
$$c = \frac{(\partial\hat{p}/\partial z)R^2}{v_0\mu_0} \quad \bar{\phi} = \frac{\phi - \phi_w}{\phi_m - \phi_w}, \quad N_b = D_b(\phi_m - \phi_w),$$
$$N_t = \frac{D_T(\theta_m - \theta_w)}{\theta_w} \quad \Lambda = \frac{2\beta_3 v_0^2}{\mu_0 R^2},$$
$$G_r = \frac{(\theta_m - \theta_w)\,\rho_{fw} R^2(1 - \phi_w)\,g}{\mu_0 u_0},$$
$$B_r = \frac{\left(\rho_p - \rho_w\right) R^2(\phi_m - \phi_w)\,g}{\mu_0 u_0} \tag{24}$$

after dropping the bars for simplicity, we get the non-dimensional governing equations of the form

$$\frac{d\mu}{dr}\frac{dv}{dr} + \frac{\mu}{r}\frac{dv}{dr} + \mu\frac{d^2v}{dr^2} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 + 3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} = c + Pv + M^2v - G_r\theta - B_r\phi \tag{25}$$

$$\alpha\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + N_b\frac{d\theta}{dr}\frac{d\phi}{dr} + \alpha_1 N_t\left(\frac{d\theta}{dr}\right)^2 = 0 \tag{26}$$

$$N_b\left(\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr}\right) + N_t\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = 0 \tag{27}$$

subject to boundary conditions

$$v(1) = 1, \quad v(2) = \gamma\left[\frac{dv}{dr}(2)\right]$$
$$\theta(1) = 1, \quad \theta(2) = 0$$
$$\phi(1) = 1, \quad \phi(2) = 0 \tag{28}$$

where v_0, μ_0, θ_w, $\bar{\theta}$, θ_m and ϕ_m denote the reference velocity, reference viscosity, reference temperature, pipe temperature, fluid temperature and mass concentration respectively. Here P, M, G_r, B_r, N_t, N_b and γ are the porosity parameter, MHD parameter, thermophoresis diffusion constant, Brownian diffusion constant, thermophoresis parameter, Brownian motion parameter and partial slip parameter respectively. The modified pressure $\hat{p}$ is

$$\hat{p} = p_1 - \alpha_2\left(\frac{dv}{dr}\right)^2 \tag{29}$$

Solutions of non-Newtonian nanofluids

In this section, we find the series solutions of the non-linear governing equations using the OHAM for two cases, namely constant and variable viscosity.

Constant viscosity model

For the constant viscosity model, we choose

$$\mu = 1 \tag{30}$$

Choosing the linear operator L_i which is defined by

$$L_i(\varphi_i(r, p)) = \frac{\partial^2\varphi_i(r, p)}{\partial r^2} \tag{31}$$

such that

$$N_1(\varphi_i(r, p)) = \frac{1}{r}\frac{dv}{dr} + \frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 + 3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} + G_r\theta + B_r\phi - Pv - M^2v \tag{32}$$

$$N_2(\varphi_i(r, p)) = \frac{1}{r}\frac{d\theta}{dr} + \frac{N_b}{\alpha}\frac{d\theta}{dr}\frac{d\phi}{dr} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{d\theta}{dr}\right)^2 \tag{33}$$

$$N_3(\varphi_i(r,p)) = \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta}{\mathrm{d}r^2}\right) \tag{34}$$

The corresponding boundary conditions are

$$\begin{aligned} &\varphi_1(1) = 1, \quad \varphi_1(2) - \gamma\left[\frac{\mathrm{d}\varphi_1}{\mathrm{d}r}\right]_{r=2} = 0 \\ &\varphi_2(1) = 1, \quad \varphi_2(2) = 0 \\ &\varphi_3(1) = 1, \quad \varphi_3(2) = 0 \end{aligned} \tag{35}$$

The zeroth order deformation. The zeroth order deformation problems are of the following form

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}r^2} = c, \quad v_0(1) = 1, \quad v_0(2) - \gamma\left[\frac{\mathrm{d}v_0}{\mathrm{d}r}\right]_{r=2} = 0 \tag{36}$$

$$\frac{\mathrm{d}^2\theta_0}{\mathrm{d}r^2} = 0, \quad \theta_0(1) = 1, \quad \theta_0(2) = 0 \tag{37}$$

$$\frac{\mathrm{d}^2\phi_0}{\mathrm{d}r^2} = 0, \quad \phi_0(1) = 1, \quad \phi_0(2) = 0 \tag{38}$$

The first and second order deformations. First and second order problems are defined by

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}r^2} = C_1\left[\frac{1}{r}\frac{\mathrm{d}v_0}{\mathrm{d}r} + \frac{\Lambda}{r}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^3 + 3\Lambda\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2 v_0}{\mathrm{d}r^2} + G_r\theta_0 + B_r\phi_0 - Pv_0 - M^2 v_0\right] \tag{39}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2} = C_1\left[\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{N_b}{\alpha}\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\right)^2\right] \tag{40}$$

$$\frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2} = C_1\left[\frac{1}{r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{N_t}{N_b}\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta_0}{\mathrm{d}r^2}\right] \tag{41}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2 v_2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2 v_1}{\mathrm{d}r^2} &+ C_2\left\{\frac{1}{r}\frac{\mathrm{d}v_0}{\mathrm{d}r} + \frac{\Lambda}{r}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^3 \right. \\ &+ 3\Lambda\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2 v_0}{\mathrm{d}r^2} + G_r\theta_0 + B_r\phi_0 \\ &\left. - P\left[1 + \lambda\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\right]v_0 - M^2 v_0\right\} \\ &+ C_1\left\{\frac{\mathrm{d}^2 v_1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v_1}{\mathrm{d}r} + \frac{3\Lambda}{r}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}v_1}{\mathrm{d}r} \right. \\ &+ G_r\theta_1 + B_r\phi_1 3\Lambda\left[\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2 v_1}{\mathrm{d}r^2} + 2\frac{\mathrm{d}v_0}{\mathrm{d}r}\frac{\mathrm{d}v_1}{\mathrm{d}r}\frac{\mathrm{d}^2 v_0}{\mathrm{d}r^2}\right] \\ &\left. - Pv_1 - M^2 v_0\right\} \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\mathrm{d}^2\theta_2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2} &+ C_2\left\{\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{N_b}{\alpha}\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + 2\frac{\alpha_1 N_t}{\alpha}\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\theta_1}{\mathrm{d}r}\right\} \\ &+ C_1\left\{\frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\theta_1}{\mathrm{d}r} + \frac{N_b}{\alpha}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_1}{\mathrm{d}r}\right)\right\} \end{aligned} \tag{43}$$

$$\begin{aligned} \frac{\mathrm{d}^2\phi_2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2} &+ C_2\left\{\frac{1}{r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{N_t}{N_b}\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta_0}{\mathrm{d}r^2}\right\} \\ &+ C_1\left\{\frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_1}{\mathrm{d}r} + \frac{N_t}{N_b}\frac{1}{r}\frac{\mathrm{d}\theta_1}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2}\right\} \end{aligned} \tag{44}$$

respectively and so forth.

Series solutions for constant viscosity model

By using the widely applied symbolic computation software MATHEMATICA to solve equations (36) to (44), we have

$$v_0 = \frac{-4 - 2c + 2r + 3cr - cr^2 + 2\gamma + 3c\gamma - 4cr\gamma + cr^2\gamma}{2(1+\gamma)} \tag{45}$$

$$\theta_0 = 2 - r \tag{46}$$

$$\phi_0 = 2 - r \tag{47}$$

$$v_1 = \frac{1}{24(1+\gamma)}\left[A_{11} + rA_{12} + r^2 A_{13} + r_{14}^3 A_{14} + r^4 A_{15}\right] \tag{48}$$

$$\theta_1 = \frac{1}{2\alpha}\left[2C_1 N_b - 3C_1 N_b r + C_1 N_b r^2 - 4C_1\alpha\ln 2 + 4C_1\alpha r\ln 2 - 2C_1\alpha\ln r\right] \tag{49}$$

$$\phi_1 = \frac{2[-2C_1 N_t\ln 2 + 2C_1 r N_t\ln 2 - C_1 N_t r\ln r]}{N_b} \tag{50}$$

$$v_2 = \left[A_{21} + rA_{22} + r^2 A_{23} + r^3 A_{24} + r^4 A_{25} + r^5 A_{26} + r^6 A_{27}\right] \tag{51}$$

$$\begin{aligned} \theta_2 = \frac{(-1+r)}{6}&\left[-6C_1\alpha\{N_b(r-2) + \alpha\ln 16\}\right] \\ &-6C_2\alpha\left[N_b(r-2) + \alpha\ln 16\right] \\ &+ C_1^2\left[N_b^2(6 - 7r + 2r^2)\right. \\ &\left. + 3N_b\left(\begin{matrix} -2N_t(r-2)\,\alpha_1 \\ + \alpha(6 - 12\ln 2 + r(\ln 16 - 3)) \end{matrix}\right)\right] \\ &+ 6\alpha\left[\begin{matrix} N_t(r - 2 + r\ln 16 - \alpha_1\ln 16) \\ -\alpha\left(2(\ln 2)^2 + \ln 16 - \ln 4\ln 16\right) \end{matrix}\right] \\ &\frac{6r}{12\alpha}\left[\begin{matrix} -2C_1\alpha - 2C_2\alpha + C_1^2\left(\begin{matrix} N_b(r-3) + 2N_t(r-\alpha_1) \\ + \alpha(\ln 16 - 2) \end{matrix}\right) \\ \ln r + 6C_1^2 r\alpha^2(\ln r)^2 \end{matrix}\right] \end{aligned} \tag{52}$$

$$\phi_2 = \frac{N_t}{2N_b^2}\begin{bmatrix} 8C_1N_b(r-1)\,\alpha\ln 2 + 8C_2N_b(r-1)\,\alpha\ln 2 \\ + C_1^2\begin{pmatrix} -4N_t(r-1)\,\alpha\ln 2\ln 8 + N_b^2 \\ (4+2r^2-6\ln 2) - r(-6+\ln 16) \end{pmatrix} \\ -N_b(r+1)\,\alpha\left(2(\ln r)^2 - \ln 4\ln 16 + \ln 4096\right) \end{bmatrix}$$
$$+ \frac{r}{2N_b^2}\left[-4C_1N_b\alpha - 4C_2N_b\alpha + C_1^2\begin{pmatrix} -3N_b^2 + 4N_t\alpha\ln 4 \\ + N_b\alpha(\ln 16 - 6)\end{pmatrix}\right]\ln r$$
$$-C_1^2(N_b + 2N_t)\,r\alpha(\ln r)^2 \tag{53}$$

Thus the series solution of equations (25) to (27) up to second order for constant viscosity is of the following form

$$\begin{aligned} v &= v_0 + v_1 + v_2 \\ \theta &= \theta_0 + \theta_1 + \theta_2 \\ \phi &= \phi_0 + \phi_1 + \phi_2 \end{aligned} \tag{54}$$

Variable viscosity model

For the variable viscosity model, we take

$$\mu = r \tag{55}$$

such that

$$L_i(\varphi_i(r,p)) = \frac{\partial^2 \varphi_i(r,p)}{\partial r^2} \tag{56}$$

and define the nonlinear operator as

$$N_1(\varphi_i(r,p)) = \frac{2}{r^2}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{\Lambda}{r^2}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^3 + \frac{3\Lambda}{r}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2v}{\mathrm{d}r^2} + \frac{G_r}{r}\theta + \frac{B_r}{r}\phi - \frac{P}{r}v - \frac{M^2}{r}v \tag{57}$$

$$N_2(\varphi_i(r,p)) = \frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{N_b}{\alpha}\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2 \tag{58}$$

$$N_3(\varphi_i(r,p)) = \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta}{\mathrm{d}r^2}\right) \tag{59}$$

along with boundary conditions

$$\begin{aligned} &\varphi_1(1) = 1, \quad \varphi_1(2) - \gamma\left[\frac{\mathrm{d}\varphi_1}{\mathrm{d}r}\right]_{r=2} = 0 \\ &\varphi_2(1) = 1, \quad \varphi_2(2) = 0 \\ &\varphi_3(1) = 1, \quad \varphi_3(2) = 0 \end{aligned} \tag{60}$$

The zeroth order deformation. The zeroth order problem is given by

$$\frac{\mathrm{d}^2v_0}{\mathrm{d}r^2} = c, \quad v_0(1) = 1, \quad v_0(2) = \gamma\left[\frac{\mathrm{d}v_0}{\mathrm{d}r}\right]_{r=2} \tag{61}$$

$$\frac{\mathrm{d}^2\theta_0}{\mathrm{d}r^2} = 0, \quad \theta_0(1) = 1, \quad \theta_0(2) = 0 \tag{62}$$

$$\frac{\mathrm{d}^2\phi_0}{\mathrm{d}r^2} = 0, \quad \phi_0(1) = 1, \quad \phi_0(2) = 0 \tag{63}$$

The first and second order deformation. The first order and second order problems are given by

$$\frac{\mathrm{d}^2v_1}{\mathrm{d}r^2} = C_1\left\{\frac{2}{r^2}\frac{\mathrm{d}v_0}{\mathrm{d}r} + \frac{\Lambda}{r^2}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^3 + \frac{3\Lambda}{r}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2v_0}{\mathrm{d}r^2} + \frac{G_r}{r}\theta_0 + \frac{B_r}{r}\phi_0 - \frac{P}{r}v_0 - \frac{M^2}{r}v_0\right\} \tag{64}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2} = C_1\left\{\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{N_b}{\alpha}\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\right)^2\right\} \tag{65}$$

$$\frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2} = C_1\left\{\frac{1}{r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{N_t}{N_b}\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta_0}{\mathrm{d}r^2}\right\} \tag{66}$$

and

$$\frac{\mathrm{d}^2v_2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2v_1}{\mathrm{d}r^2} + \frac{C_2}{r}\left\{\begin{matrix} \frac{2}{r}\frac{\mathrm{d}v_0}{\mathrm{d}r} + \frac{\Lambda}{r}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^3 + 3\Lambda\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2v_0}{\mathrm{d}r^2} \\ + G_{rr}\theta_0 + B_r\phi_0 - P\left[1 + \Lambda\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\right]v_0 - M^2v_0 \end{matrix}\right\}$$
$$+ \frac{C_1}{r}\left\{\begin{matrix} r\frac{\mathrm{d}^2v_1}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}v_1}{\mathrm{d}r} + \frac{3\Lambda}{r}\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}v_1}{\mathrm{d}r} + \; + G_{rr}\theta_1 + B_r\phi_1 \\ 3\Lambda\left[\left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2v_1}{\mathrm{d}r^2} + 2\frac{\mathrm{d}v_0}{\mathrm{d}r}\frac{\mathrm{d}v_1}{\mathrm{d}r}\frac{\mathrm{d}^2v_0}{\mathrm{d}r^2}\right] - \\ P\left[v_1 + \Lambda\left(2\frac{\mathrm{d}v_0}{\mathrm{d}r}\frac{\mathrm{d}v_1}{\mathrm{d}r}v_0 + \left(\frac{\mathrm{d}v_0}{\mathrm{d}r}\right)^2v_1\right)\right] - M^2v_0 \end{matrix}\right\} \tag{67}$$

$$\frac{\mathrm{d}^2\theta_2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2} + C_2\left\{\frac{\alpha}{(\alpha+\alpha_1N_t)\,r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{N_b}{(\alpha+\alpha_1N_t)}\frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right\}$$
$$+ C_1\left\{\frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2} + \frac{\alpha}{(\alpha+\alpha_1N_t)\,r}\frac{\mathrm{d}\theta_1}{\mathrm{d}r} + \frac{N_b}{(\alpha+\alpha_1N_t)}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{\mathrm{d}\theta_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_1}{\mathrm{d}r}\right)\right\} \tag{68}$$

$$\frac{\mathrm{d}^2\phi_2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2} + C_2\left\{\frac{1}{r}\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \frac{N_t}{N_b}\frac{1}{r}\frac{\mathrm{d}\theta_0}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta_0}{\mathrm{d}r^2}\right\}$$
$$+ C_1\left\{\frac{\mathrm{d}^2\phi_1}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\phi_1}{\mathrm{d}r} + \frac{N_t}{N_b}\frac{1}{r}\frac{\mathrm{d}\theta_1}{\mathrm{d}r} + \frac{\mathrm{d}^2\theta_1}{\mathrm{d}r^2}\right\} \tag{69}$$

respectively and so forth.

Series solutions for constant viscosity model

With the same constraints the series solution of equations (25) to (27) up to second order for variable viscosity is given by

$$v = 2(1+\gamma)\,[-4 - 2c + 2r + 3cr - cr^2 + 2\gamma + 3c\gamma - 4cr\gamma + cr^2\gamma]$$
$$+ \frac{1}{48(1+\gamma)^4}\left[B_{11} + rB_{12} + r^2B_{13} + r_{14}^3B_{14}\right]$$
$$+ \left[\frac{B_{21}}{r} + B_{22} + rB_{23} + r^2B_{24} + r^3B_{25} + r^4B_{26}\right] \tag{70}$$

$$
\begin{aligned}
\theta = {} & 2 - r + \tfrac{1}{2\alpha}[2C_1N_b - 3C_1N_br + C_1N_br - 4C_1\alpha\ln 2 + 4C_1r\alpha\ln 2 - 2C_1r\alpha\ln r] \\
& + \tfrac{(-1+r)}{12\alpha^2}(-6C_1\alpha(N_b(-2+r) + \alpha\ln 16) - 6c2\alpha(N_b(-2+r) + \alpha\ln 16)) \\
& + C_1^2\left[N_b^2\left(6 - 7r + 2r^2\right) + 3N_b\begin{pmatrix} -2N_t(r-2)\,\alpha_1 \\ + \alpha(6 - 12\ln 2 + r(\ln 16 - 3)) \end{pmatrix}\right] \\
& + 6\alpha\begin{bmatrix} N_t(r - 2 + r\ln 16 - \alpha_1\ln 16) \\ -\alpha(\ln 16 - \ln 4\ln 16) \end{bmatrix} \\
& + \tfrac{r}{2\alpha}\begin{bmatrix} -2C_1\alpha - 2C_2\alpha + C_1^2\begin{pmatrix} N_b(r-3) + 2N_t(r-\alpha_1) \\ + \alpha(\ln 16 - 2) \end{pmatrix} \\ C_1^2\alpha\ln r \end{bmatrix}
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
\phi = {} & 2 - r + \tfrac{2}{N_b}[-2C_1N_t\ln 2 + 2C_1N_tr\ln 2 - C_1N_tr\ln r] \\
& + \tfrac{N_t}{2N_b^2\alpha}\begin{bmatrix} 8C_1N_b(r-1)\,\alpha\ln 2 + 8C_2N_b(r-1)\,\alpha\ln 2 \\ + C_1^2\begin{pmatrix} -12N_b\alpha\ln 2 - 6 - 12N_b\alpha\ln 2^2 \\ + 2N_b^2(r - 2 + \ln 8) - 12N_t\alpha\ln 2^2 \end{pmatrix} \\ -N_b(r+1)\,\alpha\left(2(\ln r)^2 - \ln 4\ln 16 + \ln 4096\right) \end{bmatrix} \\
& + \tfrac{r\ln r}{2N_b^2}\left[-4C_1N_b\alpha - 4C_2N_b\alpha + C_1^2\begin{pmatrix} 3N_b^2 + 8N_t\alpha\ln 2 \\ + N_b\alpha(4\ln 2 + 6) \end{pmatrix}\right] \\
& + C_1^2(N_b + 2N_t)\,\alpha\ln r
\end{aligned} \tag{72}
$$

The coefficients A_{11} to A_{15}, A_{21} to A_{27}, B_{11} to B_{14}, and B_{21} to B_{26} can easily be obtained through the routine calculation.

Influence of pertinent parameters

The solution is obtained by the OHAM. The effects of the magnetohydrodynamic parameter M, porosity P and slip parameter γ on velocity for both constant and variable viscosity are shown in Figures 1 to 6. In

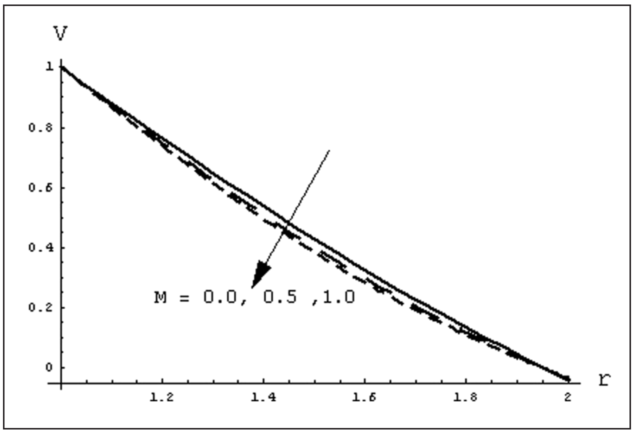

Figure 1. Effect of M on velocity profile when $N_b = N_t = 1$, $P = 0.2$ and $\gamma = 0.05$ for constant viscosity.

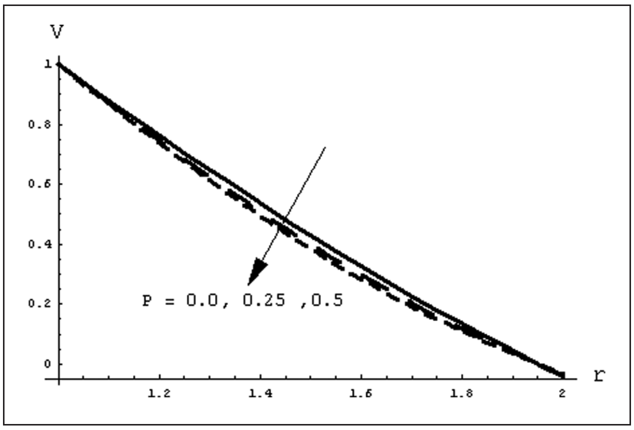

Figure 3. Effect of P on velocity profile when $N_b = N_t = 1$, $M = 0.5$ and $\gamma = 0.05$ for constant viscosity.

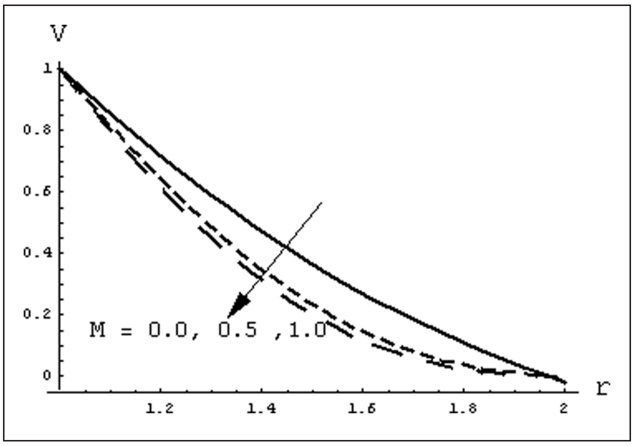

Figure 2. Effect of M on velocity profile when $N_b = N_t = 1$, $P = 0.25$ and $\gamma = 0.05$ for variable viscosity.

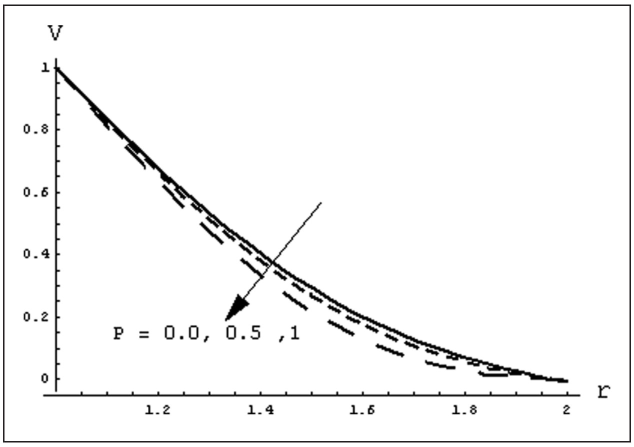

Figure 4. Effect of P on velocity profile when $N_b = N_t = 1$, $M = 0.5$ and $\gamma = 0.05$ for variable viscosity.

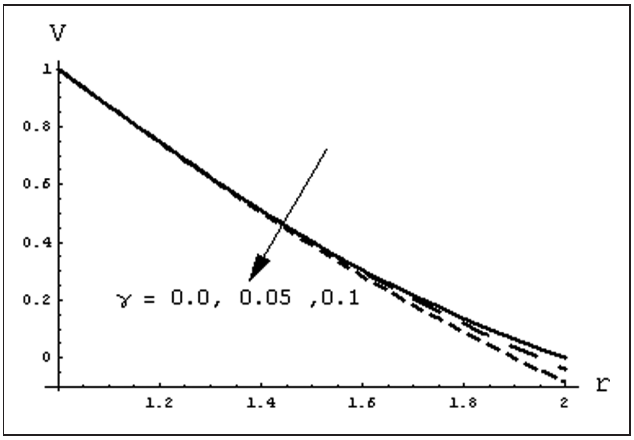

Figure 5. Effect of γ on velocity profile when $N_b = 1$, $M = 0.5$ and $P = 0.25$ for constant viscosity.

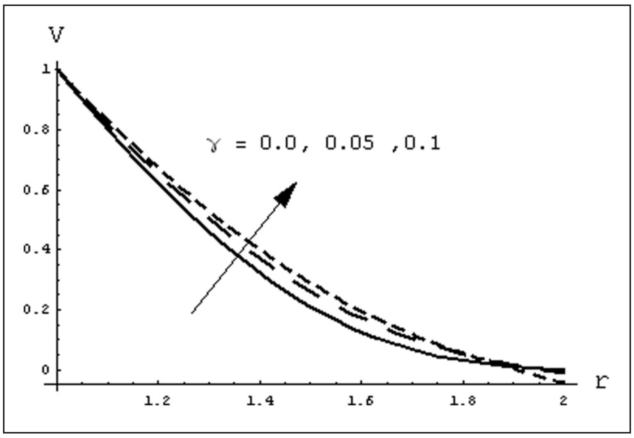

Figure 6. Effect of γ on velocity profile when $N_b = 1$, $N_t = 1$, $M = 0.5$ and $P = 0.25$ for variable viscosity.

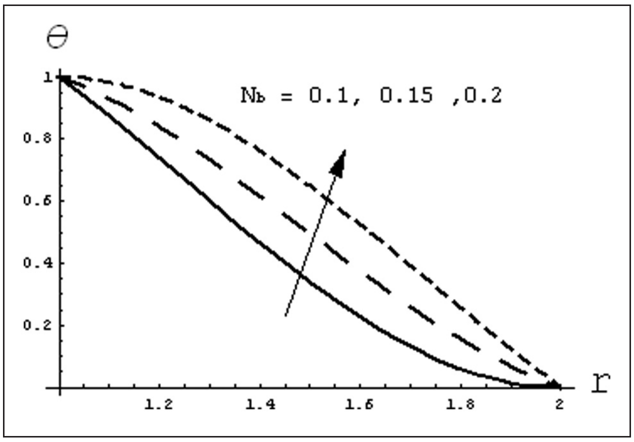

Figure 7. Effect of N_b on temperature distribution when $N_t = 0.1$.

Figures 7 to 10 the effects of N_b and N_t are shown on the velocity, nanoparticle concentration and temperature distribution respectively.

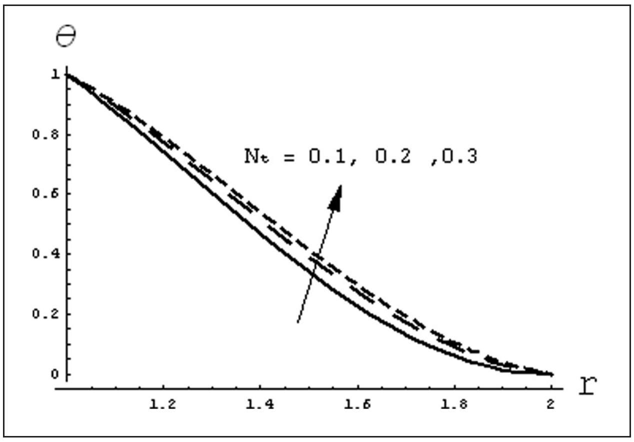

Figure 8. Effect of N_t on temperature distribution when $N_b = 0.1$.

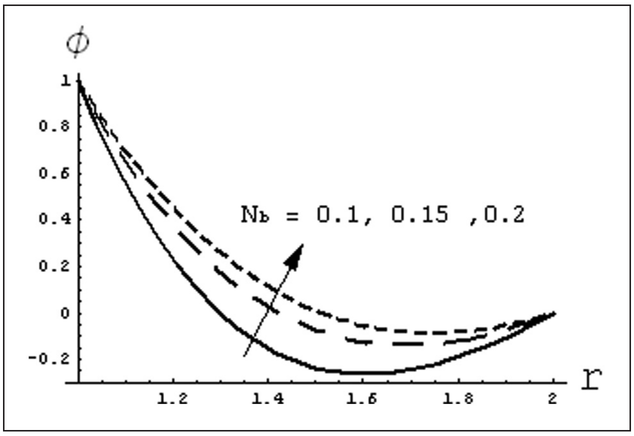

Figure 9. Effect of N_b on nanoparticle concentration when $N_t = 0.1$.

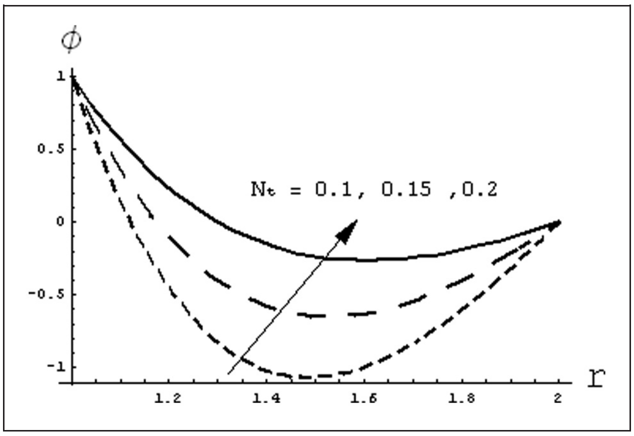

Figure 10. Effect of N_t on nanoparticle concentration when $N_b = 0.1$.

Conclusions

The effects of partial slip and MHD flow of a non-Newtonian nanofluid in a coaxial porous cylinder have been examined in this study. The resulting nonlinear differential systems are solved by employing the OHAM. In this method we control the convergence using a number j of auxiliary constants $C_1, C_2, \ldots, C_j$ which are optimally determined. Variations of MHD parameter M, porosity parameter P and slip parameter γ on velocity profile have been analyzed for constant and variable viscosity in Figures 1 to 6. To see the effects of the thermophoresis parameter and Brownian diffusion coefficient on the temperature profile, Figures 7 to 10 have been displayed. It is interesting to note that for OHAM solutions, when $\gamma = P = M = G_r = B_r = 0$ then one recovers the case of reference 37. The problem reduces to the case of reference 38 for $\gamma = P = G_r = B_r = 0$. The case of reference 39 can also be recovered by $\gamma = M = G_r = B_r = 0$. It is also worth mentioning that the presented solutions are valid for all values of sundry parameters and, to the best of our knowledge, the series solutions by OHAM for this particular model have not been presented before.

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Appendix 1

Notation

B_r	thermophoresis parameter
D_b	Brownian diffusion coefficient
D_T	thermophoretic diffusion coefficient
g	gravitational acceleration
G_r	thermophoresis diffusion constant
$\mathbf{I}$	identity tensor
k	thermal conductivity
M	MHD parameter
N_b	Brownian motion parameter
N_t	thermophoresis parameter
p_1	hydrostatic pressure
$\hat{p}$	modified pressure
P	porosity parameter
$\mathbf{T}$	Cauchy stress tensor
v, $\mathbf{V}$	velocity
v_0	reference velocity
$\alpha_i (i = 1, 2)$	material constant
$\beta_j (j = 1 - 3)$	material constant
β_T	volumetric solutal expansion coefficient of the nanofluids
γ	partial slip parameter
θ	temperature
$\overline{\theta}$	fluid temperature
θ_m	mass concentration
θ_w	reference temperature
μ	viscosity
μ_0	reference viscosity
ρ_f	density of the base fluid
ρ_p	density of the nanoparticles
ϕ	nanoparticle volume fraction
∇	gradient operator