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Analysis of gaseous slip flow in a porous micro-annulus under local thermal non-equilibrium condition – An exact solution



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ABSTRACT

The phenomenon of rarefaction in a micro-annulus filled with a porous medium is analyzed in the slip-flow regime. A local thermal non-equilibrium (LTNE) model is utilized to represent the energy transport within the porous medium. Exact solutions are derived for both the fluid and solid temperature distributions within the annulus. Two distinct cases of the thermal boundary conditions are considered, namely a constant heat flux at the outer wall and adiabatic inner wall (Case I) and vice versa (Case II). By eliminating the temperature difference between the fluid and solid phases, the local thermal equilibrium (LTE) model is theoretically proved to be a special case of the LTNE counterpart. Analytical predictions indicate that although the rarefaction leads to a reduction in the heat transfer, the effects of other thermophysical parameters such as the Biot number, the effective thermal conductivity ratio, the porous media shape factor and the annulus aspect ratio also play an important role. The results suggest that the configuration of Case II is superior to that of Case I from the heat transfer point of view.

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1. Introduction

Advances in MEMS and micro-fluidic devices have generated an increasing demand for understanding characteristics of fluid flow and heat transfer in microchannels. Rarefied phenomena usually occur when the fluid flows through a microchannel, which is the key feature compared to that at the macroscale. In such a situation, the fluid no longer reaches the velocity or the temperature of the wall surface and therefore a slip condition for the velocity and a jump condition for the temperature should be utilized [1]. The Knudsen number (*Kn*), defined as the ratio of the molecular mean free path to the appropriate characteristic length in the order of micrometers, is a measure of the degree of rarefaction [1–5]. According to the values of *Kn*, the gas flow regimes can be classified as: (i) continuum regime ($Kn \le 10^{-3}$), (ii) slip-flow regime ($10^{-3} < Kn \le 10^{-1}$), (iii) transition regime ($10^{-1} < Kn \le 10$) and, (iv) free molecular regime (Kn > 10).

Convective heat transfer in porous media is encountered in a wide variety of industrial applications such as geothermal engineering, heat pipes, electronic cooling and solar energy collection. There are two primary models for representing heat transfer in a porous medium: LTE and LTNE. The LTNE model has gained

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http://dx.doi.org/10.1016/j.ijheatmasstransfer.2015.06.001 0017-9310/© 2015 Elsevier Ltd. All rights reserved. increased attention in recent years since the assumption of local thermal equilibrium breaks down and the temperature difference between fluid and solid phases within a porous medium is significant [6–8]. The slip-flow and heat transfer in various-shaped microchannels filled with porous media have been studied. However, most of the investigations were confined to the LTE model and only a few were contributed to the LTNE counterpart. Haddad et al. [9] numerically investigated the developing hydrodynamic and thermal behaviors of free convection gas flow in a vertical open-ended parallel-plate microchannel filled with porous media. In a subsequent work, Haddad et al. [4] numerically performed the analysis of gaseous slip flow in porous parallel-plate and circular microchannels in which hydrodynamically fully developed and thermally developing forced convection were assumed. Most recently, Buonomo et al. [5] analytically investigated the gaseous slip flow in a porous parallel-plate microchannel. The exact solutions for the fluid and solid temperatures were derived under LTNE and constant heat flux conditions and the entropy generation analysis was also conducted.

The main objective of this study is to analyze forced convection flow in a micro-annulus filled with a porous medium by investigating the heat transfer characteristics with rarefaction effects. The slip-flow regime is considered, which corresponds to Knudsen numbers ranging from 10^{-3} to 10^{-1} . Two typical cases studied by Hashemi et al. [4], namely outer wall at constant heat flux (Case I) and inner wall at constant heat flux (Case II), are investigated.

Nomenclature

a _{sf}	interfacial area per unit volume of the porous medium (m^{-1})
Bi	Biot number, $h_{\rm sf}a_{\rm sf}H^2/k_{\rm s,eff}$
Cn	specific heat of the fluid $(I \text{ kg}^{-1} \text{ K}^{-1})$
Ďa	Darcy number, K/H^2
f	Fanning friction factor, $8P/(Da Re)$
$h_{\rm sf}$	interstitial heat transfer coefficient (W $m^{-2} K^{-1}$)
Ĥ	gap between two concentric cylinders (m), $R_0 - R_i$
$k_{\rm f}$	thermal conductivity of the fluid (W m^{-1} K^{-1})
$k_{\rm feff}$	effective thermal conductivity of the fluid (W m^{-1} K ⁻¹),
1,cli	$k_{\rm feff} = \varepsilon k_f$
ks	thermal conductivity of the solid (W m ^{-1} K ^{-1})
$k_{\rm s.eff}$	effective thermal conductivity of the solid (W $m^{-1} K^{-1}$),
- , -	$k_{\rm s.eff} = (1 - \varepsilon)k_{\rm s}$
Κ	permeability (m ²)
Kn	Knudsen number, λ/H
Nu	average Nusselt number at the wall for LTNE model, $-2/[(1 + \kappa)\theta_{f,b}]$
р	pressure (Pa)
Р	dimensionless pressure
Pr	Prandtl number of the fluid
q_{w}	imposed heat flux on the wall (W m^{-2})
r	radial coordinate
R _i	inner radius of the annulus (m)
Ro	outer radius of the annulus (m)
Re	Reynolds number, $ ho u_{ m m}(2H)/\mu_{ m eff}$
Т	temperature (K)
и	fluid velocity (m s ⁻¹)

In the present work, however, the temperature difference between the fluid and solid phases (LTNE model) is introduced. Furthermore, the LTE model is shown to be a special case of the LTNE model. Finally, the effects of pertinent parameters such as the Biot number, the effective thermal conductivity ratio, the porous media shape factor, the annulus aspect ratio together with the Knudsen number are discussed.

2. Formulation

2.1. Governing equations

The problem under consideration is forced convective flow between two concentric micro cylinders (micro-annulus) filled with a porous medium, as shown in Fig. 1. The inner and outer radii of the annulus are R_i and R_o respectively, and the length is *L*. One wall is uniformly heated by a constant heat flux q_w and the other one is insulated. The rarefied gas flows along the *z*-axis. In the analysis, it is assumed that the flow is steady, incompressible and both hydraulically and thermally fully developed. All the thermophysical properties of the solid and fluid phases are temperature independent. Natural convection, dispersion, radiative and axial heat conduction are negligible. The momentum and energy equations using the local thermal non-equilibrium condition in cylindrical coordinates are expressed as

Momentum equation

$$\mu_{\rm eff}\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) - \frac{\mu}{K}u - \frac{dp}{dz} = 0 \tag{1}$$

Fluid phase energy equation

$$k_{\rm f,eff} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_{\rm f}}{\partial r} \right) + h_{\rm sf} a_{\rm sf} (T_{\rm s} - T_{\rm f}) = \rho c_{\rm p} u \frac{\partial T_{\rm f}}{\partial z}$$
(2)

um	û	dimensionless velocity
	Ζ	axial coordinate
	Greek	symbols
	α	velocity slip coefficient, $[(2 - \sigma_v)/\sigma_v]Kn$
	β	temperature jump coefficient, $[(2 - \sigma_t)/\sigma_t][2\gamma/(\gamma + 1)]$
		KN/Pr
	γ	specific neat transfer ratio
-1)	δ	$=\sqrt{(1+\kappa)Bl/\kappa}$
),	3	porosity
	η	dimensionless radial coordinate, r/H
-1\	θ	dimensionless temperature, $k_{s,eff}(T - T_w)/q_w H$
•),	κ	ratio of the effective thermal conductivity of the fluid to
		that of the solid, $k_{\rm f,eff}/k_{\rm s,eff}$
	λ	molecular mean free path (m)
	μ	dynamic viscosity of the fluid (Pa s)
del,	ρ	density of the fluid (kg m^{-3})
	$\sigma_{ m t}$	thermal accommodation coefficient
	$\sigma_{\rm v}$	tangential momentum accommodation coefficient
	ω	porous media shape factor, $\sqrt{\epsilon/Da}$
	Subscr	ipts/superscripts
	e	entrance
	eff	effective
	f	fluid phase
	m	mean value
	S	solid phase

w annulus wall subject to a constant heat flux



Fig. 1. Schematic diagram of a micro-annulus filled with a porous medium for (a) Case I: outer wall at constant heat flux and (b) Case II: inner wall at constant heat flux.

Solid phase energy equation

$$k_{\rm s,eff} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_{\rm s}}{\partial r} \right) - h_{\rm sf} a_{\rm sf} (T_{\rm s} - T_{\rm f}) = 0 \tag{3}$$

where *u* is the fluid velocity, μ is the dynamic viscosity of the fluid, $k_{\rm f}$ and $k_{\rm s}$ are the thermal conductivities of the fluid and the solid matrix, $k_{\rm f,eff}$ and $k_{\rm s,eff}$ are the effective thermal conductivities and $k_{\rm f,eff} = \varepsilon k_{\rm f}$, $k_{\rm s,eff} = (1 - \varepsilon)k_{\rm s}$, $\mu_{\rm eff} = \mu/\varepsilon$ is effective viscosity, *K* is the permeability of the porous medium, *p* is the applied pressure and $T_{\rm f}$, $T_{\rm s}$, ε , ρ and $c_{\rm p}$ are the fluid and solid temperatures, the porosity, the density and specific heat of the fluid respectively. The coupling between the two energy equations is achieved using the fluid–solid interfacial term which represents the heat transfer between the two phases via the heat transfer coefficient $h_{\rm sf}$ and the specific surface area $a_{\rm sf}$.

2.2. Boundary conditions

To solve the momentum equation for the fluid velocity, the first-order velocity slip boundary condition on both annulus walls is employed as follows [4,10]

$$u_{\rm f,w} - u_{\rm w} = \alpha \frac{\partial u}{\partial \mathbf{n}} \tag{4}$$

where

$$\alpha = \frac{2 - \sigma_{\rm v}}{\sigma_{\rm v}} H K n \tag{5}$$

in which $u_{f,w}$ denotes the fluid velocity immediately adjacent to the walls, u_w is the wall velocity and is zero for the stationary wall in this study, λ is the molecular mean free path, **n** is the outward normal vector of the wall surface, σ_v stands for the tangential momentum accommodation coefficient, *H* is the gap between two concentric cylinders such that $H = R_o - R_i$ and $Kn = \lambda/H$. It should be noted that Klinkenberg effect has been incorporated by utilizing the velocity slip coefficient α . It is also worth mentioning that small Mach number is considered and hence compressibility effects of the flowing gas becomes negligible.

Similarly, in order to solve the energy equations for the fluid and solid temperatures, the first-order temperature jump boundary condition for the wall with a constant heat flux is utilized as

$$T_{\rm f,w} - T_{\rm w} = \beta \frac{\partial T_{\rm f}}{\partial \mathbf{n}} \tag{6}$$

where

$$\beta = \frac{2 - \sigma_{\rm t}}{\sigma_{\rm t}} \frac{2\gamma}{\gamma + 1} \frac{H}{Pr} Kn \tag{7}$$

in which $T_{\rm f,w}$ denotes the fluid temperature immediately adjacent to the wall, $T_{\rm w}$ is the wall temperature which is not known *a priori* and must be obtained as part of the solution, $\sigma_{\rm t}$ is the thermal accommodation coefficient, γ is the specific heat ratio and Pr is the Prandtl number. It should be noted that $T_{\rm f,w} - T_{\rm w} = 0$ for the insulated wall due to the insulated boundary condition

$$\frac{\partial T_{\rm f}}{\partial \boldsymbol{n}} = \frac{\partial T_{\rm s}}{\partial \boldsymbol{n}} = 0 \tag{8}$$

It should also be noted that the accommodation coefficients for a free fluid may be different from that for a porous medium. Unfortunately, there are no experimental results in the literature that can be used in our study. As such the accommodation coefficient for the porous media is assumed to be the same as that for the free fluid.

From the physical point of view, the values of σ_v and σ_t vary from unity (complete accommodation, diffuse reflection) to zero (specular reflection) [11]. Generally, σ_v and σ_t , both of which depend on the surface finish, the fluid temperature and pressure, need to be determined experimentally. Among most investigations σ_v and σ_t are assumed to be equal to unity. However, experimental measurements have illustrated that both values are less than 1 [12]. As suggested by Bahrami et al. [12] and Cai et al. [13], both σ_v and σ_t are therefore assumed to be 0.85 in the present study unless otherwise stated.

In addition, two more boundary conditions at the wall with constant heat flux are required based on the work of Lee and Vafai [14] when the heated wall is very thin and has high thermal conductivity

$$q_{\rm w} = k_{\rm f,eff} \frac{\partial T_{\rm f}}{\partial \boldsymbol{n}} + k_{\rm s,eff} \frac{\partial T_{\rm s}}{\partial \boldsymbol{n}} \tag{9}$$

$$T_{\rm f} = T_{\rm s} \tag{10}$$

2.3. Hydrodynamic analysis

After employing the following dimensionless variables

$$\eta = \frac{r}{H}, \quad \eta_{i} = \frac{1}{1+\psi}, \quad \eta_{o} = \frac{\psi}{1+\psi}, \quad \psi = \frac{R_{i}}{R_{o}}, \quad P = -\frac{K}{\mu u} \frac{dp}{dz},$$
$$Da = \frac{K}{H^{2}}, \quad \omega = \sqrt{\frac{\varepsilon}{Da}}, \quad \hat{u} = \frac{u}{u_{m}}$$
(11)

the Brinkman momentum Eq. (1) together with the boundary condition given by Eq. (4) can be written in the dimensionless form

$$\frac{d^2\hat{u}}{d\eta^2} + \frac{1}{\eta}\frac{d\hat{u}}{d\eta} - \omega^2(\hat{u} - P) = \mathbf{0}$$
(12)

$$\hat{u}|_{\eta=\eta_{\rm i}} = \alpha \frac{\partial \hat{u}}{\partial r}|_{\eta=\eta_{\rm i}} \tag{13a}$$

$$\hat{u}|_{\eta=\eta_{o}} = -\alpha \frac{\partial \hat{u}}{\partial r}|_{\eta=\eta_{o}}$$
(13b)

The velocity distribution can be obtained by solving the modified Bessel Eq. (12) subject to boundary conditions given by Eqs. (13a) and (13b) resulting in

$$\hat{u} = [A_1 I_0(\omega \eta) + A_2 K_0(\omega \eta) + 1]P$$
(14)

where I_{ζ} and K_{ζ} are the ζ th order modified Bessel functions of the first and second kind respectively and A_1 , A_2 are integral constants which are expressed as

$$A_{1} = \frac{K_{0}(\omega\eta_{i}) - K_{0}(\omega\eta_{o}) + \alpha\omega[K_{1}(\omega\eta_{i}) + K_{1}(\omega\eta_{o})]}{[I_{0}(\omega\eta_{i}) - \alpha\omega I_{1}(\omega\eta_{i})][K_{0}(\omega\eta_{o}) + \alpha\omega K_{1}(\omega\eta_{o})] - [I_{0}(\omega\eta_{o}) + \alpha\omega I_{1}(\omega\eta_{o})][K_{0}(\omega\eta_{i}) + \alpha\omega K_{1}(\omega\eta_{i})]}$$
(15)

$$A_{2} = \frac{I_{0}(\omega\eta_{i}) - I_{0}(\omega\eta_{o}) + \alpha\omega[I_{1}(\omega\eta_{o}) + I_{1}(\omega\eta_{o})]}{[I_{0}(\omega\eta_{i}) - \alpha\omega I_{1}(\omega\eta_{o})][K_{0}(\omega\eta_{o}) + \alpha\omega K_{1}(\omega\eta_{o})] - [I_{0}(\omega\eta_{o}) + \alpha\omega I_{1}(\omega\eta_{o})][K_{0}(\omega\eta_{i}) + \alpha\omega K_{1}(\omega\eta_{i})]}$$
(16)

and *P* can be determined through the continuum equation $\frac{1}{A} \int_A \hat{u} dA = 1$ as

$$P^{-1} = 1 + \frac{2}{\omega(1+\psi)} \{ A_1[I_0(\omega\eta_o) - \psi I_1(\omega\eta_i)] - A_2[K_0(\omega\eta_o) - \psi K_1(\omega\eta_i)] \}$$
(17)

Substituting Eqs. (15)-(17) into Eq. (14) and rearranging the related terms will lead to a similar expression for dimensionless velocity obtained by Hashemi et al. [3].

Having obtained the velocity field, we can proceed with the calculation of the Fanning friction factor as

$$f = \frac{2H\frac{dp}{dz}}{\frac{1}{2}\rho u_{\rm m}^2} = \frac{8P}{Da\,Re} = \frac{8P}{\varepsilon\omega^2\,Re}$$
(18)

where u_m is the dimensional mean velocity and *Re* is the Reynolds number, which are respectively defined by

$$u_{\rm m} = \frac{2}{R_{\rm o}^2 - R_{\rm i}^2} \int_{R_{\rm i}}^{R_{\rm o}} ur dr, \quad Re = \frac{\rho u_{\rm m}(2H)}{\mu_{\rm eff}}$$
(19)

2.4. Heat transfer analysis

2.4.1. Case I: outer wall at constant heat flux

Adding the two energy Eqs. (2) and (3), integrating the resultant equation with respect to *r* over the entire annular cross section and considering boundary conditions given by Eqs. (8) and (9) yield the following relationship

$$\rho c_{\rm p} u_{\rm m} \frac{\partial T_{\rm f}}{\partial z} = \frac{2q_{\rm w}}{H(1+\psi)} \tag{20}$$

in which $\partial T_{f,b}/\partial z = \partial T_f/\partial z = \partial T_s/\partial z = \partial T_w/\partial z = \text{const}$ for the thermally and fully developed convection. Using the following dimensionless variables

$$\kappa = \frac{k_{\rm f,eff}}{k_{\rm s,eff}}, \quad Bi = \frac{h_{\rm sf}a_{\rm sf}H^2}{k_{\rm s,eff}}, \quad \theta_{\rm f} = \frac{k_{\rm s,eff}(T_{\rm f} - T_{\rm w})}{q_{\rm w}H}, \quad \theta_{\rm s} = \frac{k_{\rm s,eff}(T_{\rm s} - T_{\rm w})}{q_{\rm w}H}$$
(21)

where κ is the ratio of the effective fluid thermal conductivity to that of the solid, *Bi* is the Biot number representing the ratio of the conduction resistance within the solid matrix to the thermal resistance associated with the internal convective heat exchange between the solid and fluid phases. Eqs. (2), (3), (6), (8) and (10) can be rewritten in the dimensionless form as

$$\kappa \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta_{\rm f}}{\partial \eta} \right) + Bi(\theta_{\rm s} - \theta_{\rm f}) = \frac{2}{1 + \psi} \hat{u}$$
(22)

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta_{s}}{\partial\eta}\right) - Bi(\theta_{s} - \theta_{f}) = 0$$
(23)

$$\theta_{\rm f}|_{\eta=\eta_{\rm o}} = -\beta \frac{\partial \theta_{\rm f}}{\partial \eta}|_{\eta=\eta_{\rm o}} \tag{24}$$

$$\frac{\partial \theta_{\rm f}}{\partial \eta}|_{\eta=\eta_{\rm i}} = \frac{\partial \theta_{\rm s}}{\partial \eta}|_{\eta=\eta_{\rm i}} = 0 \tag{25}$$

$$\theta_{\rm f}|_{\eta=\eta_{\rm o}} = \theta_{\rm s}|_{\eta=\eta_{\rm o}} \tag{26}$$

The two dimensionless energy Eqs. (22) and (23) are added, and \hat{u} is substituted from Eq. (14) to yield the following Euler-Cauchy equation for a new variable $\kappa \theta_{\rm f} + \theta_{\rm s}$

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)(\kappa\theta_{\rm f}+\theta_{\rm s}) = \frac{2}{1+\psi}(A_1I_0(\omega\eta)+A_2I_0(\omega\eta)+1)P \qquad (27)$$

The solution of Eq. (27) can be presented as

$$\kappa\theta_{\rm f} + \theta_{\rm s} = B_1 I_0(\omega\eta) + B_2 K_0(\omega\eta) + B_3 \eta^2 + B_4 \log(\eta) + B_5 \tag{28}$$

or as

$$\theta_{\rm s} = B_1 I_0(\omega \eta) + B_2 K_0(\omega \eta) + B_3 \eta^2 + B_4 \log(\eta) + B_5 - \kappa \theta_{\rm f}$$
⁽²⁹⁾

where B_1 , B_2 and B_3 are general constants which are expressed as

$$B_1 = \frac{2}{1+\psi} \frac{1}{\omega^2} \frac{A_1}{P} \tag{30}$$

$$B_2 = \frac{2}{1+\psi} \frac{1}{\omega^2} \frac{A_2}{P}$$
(31)

$$B_3 = \frac{1}{1+\psi} \frac{1}{2P}$$
(32)

while B_4 and B_5 are integral constants which need to be determined later.

After substituting Eqs. (29) and (14) into Eq. (22) and rearranging the related terms, one can obtain the following modified Bessel equation in terms of the dimensionless fluid temperature only

$$\kappa \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta_{\rm f}}{\partial \eta} \right) - \delta^2 \theta_{\rm f} = \frac{1}{\kappa} \begin{bmatrix} B_1(\omega^2 - Bi)I_0(\omega\eta) + B_2(\omega^2 - Bi)K_0(\omega\eta) \\ -B_3Bi\eta^2 - B_4Bi\log(\eta) + (4B_3 - B_5Bi) \end{bmatrix}$$
(33)

To this end, the dimensionless energy Eqs. (22) and (23) are decoupled. Subsequently, the closed form solution of Eq. (33) subject to boundary conditions given by Eqs. (24)–(26) yields the dimensionless temperature distribution of the fluid phase as

$$\theta_{\rm f} = C_1 I_0(\delta \eta) + C_2 K_0(\delta \eta) + C_3 I_0(\omega \eta) + C_4 K_0(\omega \eta) + C_5 \eta^2 + C_6 \log(\eta) + C_7$$
(34)

where C_3 , C_4 and C_5 are general constants which are obtained as

$$C_3 = \frac{1}{\omega^2 - \delta^2} \frac{1}{\kappa} (\omega^2 - Bi) B_1 \tag{35}$$

$$C_4 = \frac{1}{\omega^2 - \delta^2} \frac{1}{\kappa} (\omega^2 - Bi) B_2 \tag{36}$$

$$C_5 = \frac{B_3}{4(1+\kappa)} \tag{37}$$

$$C_6 = \frac{B_4}{1+\kappa} \tag{38}$$

$$C_7 = \frac{1}{1+\kappa} \left[B_5 - \frac{4B_3}{\kappa \delta^2} \right] \tag{39}$$

It should be noted that C_6 and C_7 are functions of B_4 and B_5 respectively. Hence, B_4 , B_5 , C_1 and C_2 are the final integral constants which can be determined by introducing the boundary conditions given by Eqs. (24)–(26) as

$$B_4 = -\eta_i [B_1 \omega I_1(\omega \eta_i) - B_2 \omega K_1(\omega \eta_i) + 2B_3 \eta_i]$$
(40)

$$B_{5} = \frac{D_{3}E_{2} - B_{4}D_{2}E_{3} - D_{2}E_{5}}{D_{2}E_{4}} + \frac{(D_{1}E_{2} - D_{2}E_{1})(D_{3}F_{2} - B_{4}D_{2}F_{3} - D_{2}F_{4})}{D_{2}E_{4}(D_{1}F_{2} - D_{2}F_{1})}$$
(41)

$$C_1 = \frac{D_3 F_2 - B_4 D_2 F_3 - D_2 F_4}{D_1 F_2 - D_2 F_1}$$
(42)

$$C_2 = \frac{D_3 F_1 - B_4 D_1 F_3 - D_1 F_4}{D_1 F_2 - D_2 F_1}$$
(43)

(46)

1

where

$$D_1 = (1 + \kappa) I_0(\omega \eta_o) \tag{44}$$

$$D_2 = (1 + \kappa) K_0(\omega \eta_0) \tag{45}$$

$$\begin{split} D_3 &= [(1+\kappa)C_3 - B_1]I_0(\omega\eta_0) + [(1+\kappa)C_4 - B_2]K_0(\omega\eta_0) \\ &- \frac{4B_3}{\kappa\delta^2} \end{split}$$

 $E_1 = I_0(\delta\eta_o) + \beta \delta I_1(\delta\eta_o) \tag{47}$

$$E_2 = K_0(\delta\eta_o) - \beta\delta K_1(\delta\eta_o) \tag{48}$$

$$E_3 = \frac{1}{1+\kappa} \left[\log(\eta_i) + \frac{\beta}{\eta_i} \right]$$
(49)

$$E_4 = \frac{1}{1+\kappa} \tag{50}$$

$$E_{5} = C_{3}[I_{0}(\omega\eta_{o}) + \beta\omega I_{1}(\omega\eta_{o})] + C_{4}[K_{0}(\omega\eta_{o}) - \beta\omega K_{1}(\omega\eta_{o})] + C_{5}(\eta_{o} - 2\beta)\eta_{o} - \frac{4B_{3}}{\kappa(1+\kappa)\delta^{2}}$$
(51)

 $F_1 = \delta I_1(\delta \eta_i) \tag{52}$

$$F_2 = -\delta K_1(\delta \eta_i) \tag{53}$$

$$F_3 = \frac{1}{(1+\kappa)\eta_i} \tag{54}$$

$$F_4 = C_3 \omega I_1(\omega \eta_i) - C_4 \omega K_1(\omega \eta_i) + 2C_5 \eta_i$$
(55)

2.4.2. Case II: inner wall at constant heat flux

Adding the two energy Eqs. (2) and (3), integrating the resultant equation with respect to *r* over the entire annular cross section and considering boundary conditions given by Eqs. (8) and (9) lead to

$$\rho c_{\rm p} u_{\rm m} \frac{\partial T_{\rm f}}{\partial z} = \frac{2\psi q_{\rm w}}{H(1+\psi)} \tag{56}$$

Thus, the energy Eq. (22) becomes

$$\kappa \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta_{\rm f}}{\partial \eta} \right) + Bi(\theta_{\rm s} - \theta_{\rm f}) = \frac{2\psi}{1 + \psi} \hat{u} \tag{57}$$

and the boundary conditions given by Eqs. (24)-(26) change to

$$\theta_{\rm f}|_{\eta=\eta_{\rm i}} = \beta \frac{\partial \theta_{\rm f}}{\partial \eta}|_{\eta=\eta_{\rm i}} \tag{58}$$



Fig. 2. Validation procedure.

$$\frac{\partial \theta_{\rm f}}{\partial \eta}|_{\eta=\eta_{\rm o}} = \frac{\partial \theta_{\rm s}}{\partial \eta}|_{\eta=\eta_{\rm o}} = 0 \tag{59}$$

$$\theta_{\rm f}|_{\eta=\eta_{\rm i}} = \theta_{\rm s}|_{\eta=\eta_{\rm i}} \tag{60}$$

Using the solution procedure as done for Case I, the dimensionless temperature distributions for the fluid and solid phases can be obtained by solving the two energy Eqs. (57) and (23) subject to the boundary conditions given by Eqs. (58)–(60). The analytical solutions have the same form given by Eqs. (34) and (29). However, the involved constants B_1 , B_2 , B_3 , D_1 , D_2 , D_3 , E_1 , E_2 , E_3 , E_5 , F_1 , F_2 , F_3 and F_4 should be respectively replaced by

$$B_1' = \frac{2\psi}{1+\psi} \frac{1}{\omega^2} \frac{A_1}{P}$$
(61)

$$B_2' = \frac{2\psi}{1+\psi} \frac{1}{\omega^2} \frac{A_2}{P}$$
(62)

$$B'_3 = \frac{\psi}{1+\psi} \frac{1}{2P}$$
 (63)

$$D'_1 = (1+\kappa)I_0(\omega\eta_i) \tag{64}$$

$$D_2' = (1 + \kappa) K_0(\omega \eta_i) \tag{65}$$

$$D'_{3} = [(1+\kappa)C_{3} - B'_{1}]I_{0}(\omega\eta_{i}) + [(1+\kappa)C_{4} - B'_{2}]K_{0}(\omega\eta_{i}) - \frac{4B'_{3}}{\kappa\delta^{2}}$$
(66)

$$E_1' = I_0(\delta\eta_i) + \beta \delta I_1(\delta\eta_i) \tag{67}$$

$$E_2' = K_0(\delta\eta_i) - \beta \delta K_1(\delta\eta_i) \tag{68}$$

$$E'_{3} = \frac{1}{1+\kappa} \left[\log(\eta_{o}) + \frac{\beta}{\eta_{o}} \right]$$
(69)

$$E'_{5} = C_{3}[I_{0}(\omega\eta_{i}) + \beta\omega I_{1}(\omega\eta_{i})] + C_{4}[K_{0}(\omega\eta_{i}) - \beta\omega K_{1}(\omega\eta_{i})] + C_{5}(\eta_{i} - 2\beta)\eta_{i} - \frac{4B'_{3}}{\kappa(1+\kappa)\delta^{2}}$$
(70)

$$F_1' = \delta I_1(\delta \eta_0) \tag{71}$$

$$F_2' = -\delta K_1(\delta \eta_0) \tag{72}$$

$$F_3' = \frac{1}{(1+\kappa)\eta_0} \tag{73}$$

$$F'_4 = C_3 \omega I_1(\omega \eta_o) - C_4 \omega K_1(\omega \eta_o) + 2C_5 \eta_o$$
(74)

2.5. One equation model

As expected, taking the limit of $Bi \rightarrow \infty$, the solution of the LTNE model may reduce to that of the LTE counterpart [5]. However, an alternative approach is employed for the solution of the LTE (one equation) model [7,8]. After adding Eqs. (22) and (23) for Case I or adding Eqs. (57) and (23) for Case II and assuming that the fluid and solid temperatures being equal, $\theta_f = \theta_s = \theta$, the single energy equation and corresponding boundary conditions can be written as

$$(1+\kappa)\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) = \frac{2}{1+\psi}\hat{u}$$
(75)

$$\theta|_{\eta=\eta_{o}} = -\beta \frac{\partial \theta}{\partial \eta}|_{\eta=\eta_{o}}$$
(76)

$$\frac{\partial \theta}{\partial \eta}|_{\eta=\eta_{i}} = 0 \tag{77}$$

for Case I and

$$(1+\kappa)\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) = \frac{2\psi}{1+\psi}\hat{u}$$
(78)

$$\theta|_{\eta=\eta_{i}} = \beta \frac{\partial \theta}{\partial \eta}|_{\eta=\eta_{i}}$$
⁽⁷⁹⁾

$$\frac{\partial \theta}{\partial \eta}|_{\eta=\eta_0} = 0 \tag{80}$$

for Case II.

The dimensionless temperature distributions for both Cases I and II are found to be

$$\theta = G_1 + G_2 \log(\eta) + \frac{2\Lambda P}{(1+\kappa)(1+\psi)\omega^2} [A_1 I_0(\omega\eta) + A_2 K_0(\omega\eta) + \frac{1}{4}\omega^2 \eta^2]$$
(81)

where $\Lambda = 1$ for Case I and $\Lambda = \psi$ for Case II and the constants G_1 , G_2 are given by

$$G_1 = -\frac{P}{(1+\kappa)(1+\psi)\omega} [2A_1I_1(\omega\eta_i) - 2A_2K_1(\omega\eta_i) + \omega\eta_i]\eta_i \qquad (82)$$

$$G_{2} = -G_{1}[\log(\eta_{o}) + \frac{\beta}{\eta_{o}}] - \frac{p}{2(1+\kappa)(1+\psi)\omega} \{4A_{1}[I_{0}(\omega\eta_{o}) + \beta I_{1}(\omega\eta_{o})] + 4A_{2}[K_{0}(\omega\eta_{o}) - \beta K_{1}(\omega\eta_{o})] + \omega^{2}(\eta_{o} + 2\beta)\eta_{o}\}$$

$$(83)$$

for Case I and G_1 , G_2 should be replaced by G'_1 , G'_2 for Case II:

$$G_1' = -\frac{P\psi}{(1+\kappa)(1+\psi)\omega} [2A_1I_1(\omega\eta_o) - 2A_2K_1(\omega\eta_o) + \omega\eta_o]\eta_o$$
(84)

$$\begin{aligned} G_2' &= -G_1' \left[\log(\eta_i) - \frac{\beta}{\eta_i} \right] \\ &- \frac{P\psi}{2(1+\kappa)(1+\psi)\omega^2} \left\{ 4A_1 [I_0(\omega\eta_i) - \beta I_1(\omega\eta_i)] + 4A_2 [K_0(\omega\eta_i) + \beta K_1(\omega\eta_i)] \right. \\ &+ \omega^2 (\eta_i - 2\beta)\eta_i \right\} \end{aligned}$$

$$(85)$$

2.6. Heat transfer performance

The Nusselt number based on the fluid temperature can be used for characterizing the heat transfer results. The wall heat transfer coefficient for the local thermal non-equilibrium model is obtained from

$$h_{\rm w} = \frac{q_{\rm w}}{T_{\rm w} - T_{\rm f,b}} \tag{86}$$

From the analytical solutions for the velocity and temperature distributions, the average Nusselt number on the wall with the constant heat flux, being the primary quantity of interest in heat transfer calculations, is determined based on the overall thermal equivalent conductivity, $k_{eq} = k_{f.eff} + k_{s.eff}$, as

$$Nu = \frac{2Hh_{\rm w}}{k_{\rm eq}} = -\frac{2Hq_{\rm w}}{k_{\rm eq}(T_{\rm w} - T_{\rm f,b})} = -\frac{2k_{\rm s,eff}}{k_{\rm eq}\theta_{\rm f,b}} = -\frac{2}{(1+\kappa)\theta_{\rm f,b}}$$
(87)

where $\theta_{\rm f,b}$ denotes the dimensionless bulk mean fluid temperature and is defined by

$$\theta_{\rm f,b} = \frac{2}{\eta_{\rm o}^2 - \eta_{\rm i}^2} \int_{\eta_{\rm i}}^{\eta_{\rm o}} \hat{u} \theta_{\rm f} \eta d\eta \tag{88}$$

3. Results and discussion

3.1. Validation

In all the following calculations, $\gamma = 1.4$ and Pr = 0.707 are assumed unless otherwise noted. The heat transfer performance, represented by the Nussult number, is evaluated with the dimensionless parameters in the analytical solution. To confirm the validity of the proposed exact solution, three limiting cases: the LTE pipe ($Bi \rightarrow \infty$ and $\psi \rightarrow 0$) [2], the LTE annulus ($Bi \rightarrow \infty$) [4] and the LTNE parallel-plate channel ($\psi \rightarrow 1$) [5] are provided in Fig. 2. As shown in Fig. 3, the present analytical solutions, in the respective limiting cases, agree very well with the results given in these cited three references.

3.2. Parametric analysis

Fig. 4 shows the variation of *f* Re and the velocity slip u_{slip} with the porosity ε , the porous media shape factor ω or the annulus aspect ratio ψ together with the Knudsen number *Kn*. As seen from Fig. 4, f Re decreases with an increase in Kn. This is because an increase in Knudsen number would lead to an enhancement in Re due to an increase in the flow velocity and a reduction in f due to the gas rarefaction at the wall. For $\omega < 1$, the wall friction (Brinkman shear stress term) predominates in the competition with the Darcy viscous drag while for ω > 20, the impacts of both Re and f on their product would be close to each other which makes it independent of the Knudsen number. Furthermore, a decrease in ε or an increase in ω leads to an enhancement in *f* Re while the value of ψ has little effect on *f* Re. A decrease in the porosity or an increase in the porous media shape factor would result in an increase in the volume fraction of solid matrix, which enhances the flow resistance and in turn increases the wall friction. The dimensionless velocity slip \hat{u}_{slip} for Cases I and II should be the same due to their independence of the thermal boundary conditions. It is observed that the velocity slip at the outer wall is greater than that at the inner wall.

Fig. 5 shows the variation of the dimensionless temperature jump θ_{jump} with the Biot number Bi, the effective thermal conductivity ratio κ , the porous media shape factor ω and the aspect ratio ψ together with the Knudsen number Kn. An increase in Kn or ω leads to an increase in the temperature jump T_{jump} which is proportional to the negative value of θ_{jump} . However, the parameters Bi, κ and ψ exhibit opposite effects. It is seen that the temperature jump at the heated wall for Case I is greater than that for Case II.

Fig. 6 displays the effect of the Biot number *Bi* on the Nusselt number *Nu* for κ =10⁻⁴, 10⁻², 1 and 10². As expected, an increase in *Bi* leads to an enhancement in the Nusselt number. Increasing *Bi* translates to an increase in the interstitial heat transfer which mainly occurs at the walls. The asymptotic values can be found in Fig. 6 for larger values of *Bi*. As mentioned before, the LTNE model would converge to the LTE counterpart as $Bi \rightarrow \infty$. It is also seen that the LTE values is larger than the LTNE ones because the former usually overestimate the heat transfer rate [5]. It can also be seen that the values of *Nu* become independent of *Bi* for larger values of κ , as shown in Fig. 6(d). Under the same thermophysical parameters, the values of *Nu* evaluated from Case II are greater than that obtained for Case I, which means that the inner wall at a constant heat flux is more helpful for the heat transfer enhancement as compared to the outer wall at a constant heat flux.

Fig. 7 depicts the impact of the effective thermal conductivity ratio κ on the Nusselt number *Nu* for $Bi = 10^{-2}$, 1, 10² and 10⁴. As can be seen, an increase in κ yields an enhancement in the Nusselt number. The asymptotic values of *Nu* appear when $\kappa > 10^2$ regardless of the magnitude of the Biot number.



Fig. 3. Comparison of the present results of Nusselt number with those of available three limiting cases.



Fig. 4. Effect of Knudsen number on the Fanning friction factor and the velocity slip for different (a) porosities, (b) porous media shape factors, and (c) Knudsen numbers.



Fig. 5. Effect of Knudsen number on the temperature jump for different (a) Biot numbers, (b) effective thermal conductivity ratios, (c) porous media shape factors, and (d) aspect ratios.



Fig. 6. Effect of Biot number on the Nusselt number distribution for different Knudsen numbers with (a) $\kappa = 10^{-4}$, (b) $\kappa = 10^{-2}$, (c) $\kappa = 1$, and (d) $\kappa = 10^{2}$.



Fig. 7. Effect of the effective thermal conductivity ratio on the Nusselt number distribution for different Knudsen numbers with (a) $Bi = 10^{-2}$, (b) Bi = 1, (c) $Bi = 10^{2}$, and (d) $Bi = 10^{4}$.



Fig. 8. Effect of the porous media shape factor on the Nusselt number distribution for different Knudsen numbers with (a) $\kappa = 10^{-4}$, (b) $\kappa = 10^{-2}$, (c) $\kappa = 1$, and (d) $\kappa = 10^{2}$.



Fig. 9. Effect of the aspect ratio on the Nusselt number distribution for different Knudsen numbers with (a) $\omega = 10^{-2}$, (b) $\omega = 1$, (c) $\omega = 10$, and (d) $\omega = 10^2$.



Fig. 10. Effect of the temperature jump on the Nusselt number distribution with (a) $\omega = 10^{-2}$, (b) $\omega = 1$, (c) $\omega = 10$, and $\omega = 10^{2}$.

Physically, a larger value of κ implies a higher contribution of forced convection currents at the walls. As illustrated in Fig. 7(d), the Nusselt number dependence on κ is diminished for higher values of *Bi* since the behavior of the LTNE model approaches that of the LTE model. Again, the heat transfer performance for Case II is better than that for Case I.

Fig. 8 displays the effect of the porous media shape factor ω on the Nusselt number *Nu* for $\kappa = 10^{-4}$, 10^{-2} , 1 and 10^2 . For a given value of *Kn*, the Nusselt number is not affected much when $\omega < 1$ while the Nusselt number changes slightly when $\omega > 1$. As was mentioned before, the rarefaction decreases the heat transfer, i.e. increasing *Kn* leads to a reduction in *Nu*. Meanwhile, Case II results in a larger Nusselt number than Case I.

Fig. 9 illustrates the variation of the Nusselt number *Nu* with the annulus aspect ratio ψ for $\omega = 10^{-2}$, 1, 10 and 10². As shown in Fig. 9, the Nusselt number for Case I increases slightly with increasing aspect ratio while for Case II it drops down sharply with an increase in the aspect ratio. The larger the aspect ratio, the closer the values of *Nu* for Cases I and II. As expected, the *Nu* value for Case I would be equal to that for Case II when the aspect ratio approaches 1. It is known that taking the limit of $\psi \rightarrow 1$, both Cases I and II reduce to the parallel-plate channel subject to a constant heat flux at one surface and an insulated boundary at the other.

As seen from the aforementioned parametric study, an increase in the Knudsen number results in a reduction in the Nusselt number. Since both velocity slip and temperature jump conditions involve the Knudsen number, it will be interesting to find out if either one or both could weaken the heat transport in the porous media. Fig. 10 displays the effect of the temperature jump on the Nusselt number for $\omega = 10^{-2}$, 1, 10 and 10². When the temperature jump condition is not taken into account, only the velocity slip condition ($\alpha \neq 0$, $\beta = 0$) is considered and vice versa. It can be seen that the velocity slip enhances the Nusselt number whereas the temperature jump has an opposite effect. Similar results were also reported by Tunc and Bayazitoglu [15] for microtubes. Also, previously published analytical works on slip-flow convection [16,17] can be noted in here. As compared to the velocity slip, the temperature jump dominates the heat transfer process, which leads to a reduction in the heat transfer. The results obtained under both the velocity slip and temperature jump conditions are also plotted in this figure. It should be noted that the results on the ordinate axis correspond to the no-slip regime solutions.

4. Conclusions

A theoretical investigation of the heat transfer characteristics for the forced convective gaseous flow through a micro-annulus filled with a porous material is presented in this work. The Brinkman momentum equation is employed to describe the fluid flow within the porous medium and the two-energy equation model is utilized due to the inherent coupling of the fluid and solid phases. To this end, exact solutions are obtained for both the fluid and solid temperature distributions under the LTNE condition. Utilizing the presented exact solutions, the Nusselt number is obtained as a function of five pertinent parameters including the Biot number Bi, the effective thermal conductivity ratio κ , the porous media shape factor ω , the annulus aspect ratio ψ , and the Knudsen number *Kn*. Two distinct cases (Cases I and II) are considered, which are validated by three limiting cases available in the literature. Our investigation establishes that the rarefaction reduces the heat transfer within the porous medium, in which the velocity slip and the temperature jump have opposite effects. Also, the configuration of Case II is shown to create more heat transfer enhancement within the porous medium as compared to that of Case I.

Conflict of interest

None declared.

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