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# The effect of radiation on free convection flow of fluid with variable viscosity from a porous vertical plate

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**Abstract** — The effect of thermal radiation on the natural convection flow along a uniformly heated vertical porous plate with variable viscosity and uniform suction velocity was investigated numerically. The fluid considered in this study is of an optically dense viscous incompressible fluid of temperature-dependent viscosity. The laminar boundary layer equations governing the flow are shown to be nonsimilar. The governing equations are analyzed using a variety of methods: (i) a series solution for small values of $\xi$ (a scaled streamwise coordinate depending on the transpiration); (ii) an asymptotic solution for large values of $\xi$; and (iii) a full numerical solution using the Keller box method. The solutions are expressed in terms of the local shear stress and the local heat transfer rate. The working fluid is taken to have Prandtl number $Pr = 1$, and the effects of varying the viscosity variation parameter, $\gamma$, the radiation parameter, $R_d$, and the surface temperature parameter, $\theta_w$, are discussed. © 2001 Éditions scientifiques et médicales Elsevier SAS

**radiation / free convection / variable viscosity / porous vertical plate**

## Nomenclature

| | | |
|---|---|---|
| $a$ | Rosseland mean absorption coefficient | |
| $C_{fx}$ | local skin friction coefficient | |
| $g$ | acceleration due to gravity | $m{\cdot}s^{-2}$ |
| $Gr$ | Grashof number | |
| $Nu_x$ | local Nusselt number | |
| $Pr$ | Prandtl number | |
| $q_r$ | the component of radiative flux in $y$ direction | $w{\cdot}m^{-2}$ |
| $R_d$ | radiation parameter | |
| $T$ | fluid temperature | K |
| $T_\infty$ | ambient fluid temperature | K or °C |
| $T_w$ | porous plate temperature | K or °C |
| $u, v$ | reference velocity components in $x$ and $y$ directions | $m{\cdot}s^{-1}$ |
| $\bar{u}, \bar{v}$ | nondimensional velocity components in $x$ and $y$ directions | |
| $V_0$ | transpiration velocity of fluid through the surface of the plate velocity | $m{\cdot}s^{-1}$ |
| $x, y$ | Cartesian coordinates | m |

*Greek symbols*

| | | |
|---|---|---|
| $\alpha$ | thermal diffusivity | $m^2{\cdot}s^{-1}$ |
| $\beta$ | thermal expansion coefficient | $K^{-1}$ |
| $\rho$ | density of the ambient fluid | $kg{\cdot}m^{-3}$ |
| $k$ | thermal conductivity | $W{\cdot}m^{-1}{\cdot}K^{-1}$ |
| $\gamma$ | viscosity variation parameter | |
| $\mu$ | absolute viscosity | $kg{\cdot}m^{-1}{\cdot}s^{-1}$ |
| $\mu_f$ | absolute viscosity at the film temperature | $kg{\cdot}m^{-1}{\cdot}s^{-1}$ |
| $\psi$ | stream function | $m^2{\cdot}s^{-1}$ |
| $\sigma$ | Stefan–Boltzmann constant | $W{\cdot}m^{-2}{\cdot}K^{-4}$ |
| $\sigma_s$ | scattering coefficient | |
| $\theta$ | dimensionless temperature function | |
| $\theta_w$ | temperature parameter | |
| $\xi$ | scaled streamwise coordinate | m |

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## 1. INTRODUCTION

Convective boundary-layer flows are often controlled by injecting or withdrawing fluid through a porous bounding heated surface. This can lead to enhanced heating or cooling of the system and can help to delay the transition from laminar to turbulent flow. Previous work on the effects of blowing and suction on free convection boundary layers had been confined to cases with a prescribed wall temperature. The case of uniform suction and blowing through an isothermal vertical wall was treated first by Sparrow and Cess [1]. They obtained a solution in series form valid near the leading edge. The effects of uniform blowing and suction on the natural convection boundary layer from a vertical plate were discussed by Merkin [2]. In this study, a numerical solution of the full boundary layer equations was obtained for both blowing and suction. The next-order corrections to the boundary layer solution for this problem were obtained by Clarke [3]. Boussinesq approximation was not invoked in this investigation at any stage in describing the flow field. The effect of strong suction and blowing from general body shape, which admit a similarity solution, was investigated by Merkin [4]. Asymptotic solutions for both strong blowing and suction cases were derived and compared well with the numerical results. A transformation of the boundary layer equations for natural convection past a vertical plate with an arbitrary blowing and wall temperature variations was studied by Vedhanayagam et al. [5]. The case of a heated isothermal horizontal surface with transpiration was discussed in some detail first by Clarke and Riley [6, 7], and then recently by Lin and Yu [8].

In all the above studies, the viscosity of the fluid was assumed to be uniform throughout the flow regime. However, it is known that this physical property may change significantly with temperature. Accordingly, Gary et al. [9] and Mehta and Sood [10] have concluded that when this effect is included, the flow characteristics substantially change compared to the constant viscosity case. Recently, Kafoussius and Williams [11] and Kafoussias and Rees [12] have investigated the effect of the temperature-dependent viscosity on the mixed convection flow past a vertical flat plate in the region near the leading edge using the local nonsimilarity method. In these studies, they concluded that when the viscosity of a fluid is sensitive to temperature variations, the effect of temperature-dependent viscosity has to be taken into consideration, otherwise considerable errors may occur in the characteristics of the heat transfer process. Hossain and Kabir [13] have investigated the natural convection flow from a vertical wavy surface. Hossain and Munir [14] investigated the mixed convection flow from a vertical flat plate for a temperature-dependent viscosity. In the studies [13, 14] the viscosity of the fluid has been considered to be inversely proportional to a linear function of temperature. In all the above studies were confined without any effect of radiation.

The interaction of natural convection with thermal radiation has increased greatly during the last decade due to its importance in many practical applications. Radiation effects on the free convection flow are important in context of space technology and processes involving high temperature and very little is known about the effects of radiation on the boundary layer flow of a radiating fluid past a body. Saundalgekar and Takhar [15], first, studied the effect of radiation on the natural convection flow of a gas past a semi-infinite flat plate using the Cogley–Vincentine–Giles equilibrium model (Cogley et al. [16]). Later, Hossain and Takhar [17] analyzed the effect of radiation using the Rosseland diffusion approximation which leads to nonsimilar solutions for the forced and free convection flow of an optically dense viscous incompressible fluid past a heated vertical plate with uniform free stream and uniform surface temperature. In this analysis consideration has been given to gray gases that emit and absorb but do not scatter thermal radiation. Considering the Rosseland diffusion approximation, Hossain et al. [18] have investigated the effect of radiation on natural convection flow of an optically thick viscous incompressible flow past a heated vertical porous plate with a uniform surface temperature and a uniform rate of suction.

In the present study it is proposed to investigate the effect of temperature-dependent viscosity on the natural convection flow of an optically thick viscous incompressible fluid from a vertical porous isothermal flat plate, where radiation effect is included by invoking Rosseland diffusion approximation. In formulating the equations governing the flow, a semi-empirical formula for the viscosity proportional to a linear function of temperature has been used, following Carey and Mollendorf [19]. Using a set of transformations first introduced by Hossain et al. [18], the boundary layer equations are found to admit nonsimilarity form. These equations are solved using a series solution valid for small values of $\xi$, a scaled streamwise coordinate, and an asymptotic theory for large values of $\xi$. Solutions at intermediate values of $\xi$ are obtained using the Keller box method of Keller [20]. Solutions are presented in both graphical and tabular form and are given in terms of the local skin friction and the local heat transfer rate. Various values of the viscosity variation parameter, $\gamma$, radiation parameter, $R_d$, and temperature parameter, $\theta_w$, are considered for fluid having Prandtl number $Pr = 1.0$.

## 2. MATHEMATICAL FORMULATION

Consider a semi-infinite porous plate maintained at a uniform temperature $T_w$ is placed vertically in an infinite expanse of stagnant radiating fluid at constant temperature $T_\infty$. The following assumptions are made in this investigation:

(1) the fluid physical properties are assumed constant except for the fluid viscosity, which varies linearly with the fluid temperature and the density variation in the body force term in the momentum equation where the Boussinesq approximation is invoked,

(2) the fluid is assumed to be a gray, emitting and absorbing, but nonscattering medium, and

(3) the optically thick radiation limit is considered in the present study where the radiative heat flux term can be simplified by using the Rosseland approximation.

In addition, the radiative heat flux term in $x$ direction is considered negligible in comparison with that in the $y$ direction. The coordinates system and the flow configuration are shown in *figure 1*.

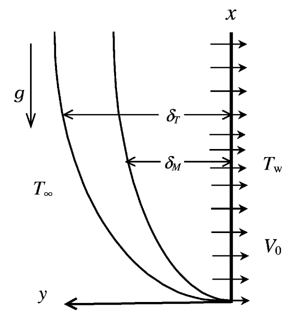


**Figure 1.** The coordinate system and the flow configuration.

Under these assumptions, the conservation equations for steady two-dimensional laminar boundary layer flow problem under consideration can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\,\frac{\partial u}{\partial \bar{x}}+v\,\frac{\partial u}{\partial y}=\frac{1}{\rho}\,\frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right)+g\beta(T-T_\infty) \tag{2}$$

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial y^2}-\frac{1}{k}\,\frac{\partial q_r}{\partial y}\right) \tag{3}$$

where $u$ and $v$ are the velocity components in the $x$ and $y$, directions, respectively, $\rho$ the density of ambient fluid, $g$ the acceleration due to gravity, $\beta$ the coefficient of thermal expansion, $k$ the coefficient of thermal conductivity, $T$ the temperature of the fluid, and $q_r$ the component of radiative flux in $y$ direction. The absolute viscosity $\mu$ is assumed to vary with temperature according to a general functional form $\mu=\mu_f s(T)$, where $\mu_f$ is the absolute viscosity at the film temperature $T_f$ and, therefore, $s(T_f)=1$. This form is chosen to allow definition of the stream function based on the absolute viscosity at the film temperature. For liquids, all transport properties vary with temperature. However, for many liquids, among them petroleum oils, glycerin, glycols, silicone fluids and some molten salt, the percent variation of absolute viscosity with temperature is much more than that of the other properties. Under the above conditions, an analysis incorporating the above assumptions and describing the momentum and thermal transport within the flow field are more accurate than the usual assumption of constant properties evaluated at some reference temperature. It should be mentioned here that there are some fluids for which properties other than $\mu$ vary strongly with temperature. In particular, water and methyl alcohol exhibit strong variation of both $\mu$ and $\beta$. The analysis presented here is not applicable to these liquids since we are considering only the variation of the absolute viscosity as a function of temperature. However, for the case of an isothermal surface (in an unstratified ambient fluid), the variation of the absolute viscosity with temperature takes the form $\mu=\mu_f S(\theta)$, where $\theta$ is the dimensionless temperature in the boundary layer defined in equation (7), such that $S(1/2)=1$. A wide variety of functional forms of $S(\theta)$ satisfying this requirement was investigated in the literature such as algebraic expressions, power series, exponential forms, etc. Following Carey and Mollendorf [19], the simplest form of the absolute viscosity is used in this investigation as follows:

$$\mu=\mu_f\left[1+\frac{1}{\mu_f}\left(\frac{d\mu}{dT}\right)_f(T-T_\infty)\right] \tag{4}$$

This simple form amounts to a linear variation of the absolute viscosity with temperature, with the slope $d\mu/dT$, evaluated at film temperature. The assumed linear variation of viscosity with temperature gives rise to a new parameter $\gamma$ defined by

$$\gamma=\frac{1}{\mu_f}\left(\frac{d\mu}{dT}\right)_f(T_w-T_\infty) \tag{5}$$

The quantity $q_r$ on the right-hand side of equation (3) represents the radiative heat flux in the $y$ direction. In order to reduce the complexity of the problem and to provide a means of comparison with future studies, a more detailed representation for the radiative heat flux, for optically thick radiation limit, is considered in the present analysis. Thus, the radiative heat flux term in the energy equation is simplified by utilizing the Rosseland diffusion approximation (Sparrow and Cess [21]) for an optically thick boundary layer as follows:

$$q_r = -\frac{4\sigma}{3k(a+\sigma_s)}\frac{\partial T^4}{\partial y} \tag{6}$$

where $\sigma$ is the Stefan–Boltzmann constant, $a$ is the Rosseland mean absorption coefficient and $\sigma_s$ is the scattering coefficient. This approximation is valid at points optically far from the bounding surface, and is good only for intensive absorption, that is, for an optically thick boundary layer.

The boundary conditions to be satisfied are

$$\begin{aligned} &y = 0: \quad u = 0,\ v = -V_0,\ T = T_\infty \\ &y \to \infty: \quad u \to 0,\ T \to T_\infty \end{aligned} \tag{7}$$

where $V_0$ represents the transpiration velocity of fluid through the surface of the plate. Here, for suction or withdrawal of fluid the transpiration velocity, $V_0$, is negative, whereas for blowing or injection of fluid $V_0$ is positive.

Near the leading edge, the boundary layer is very much like that of the free convection boundary layer in the absence of suction. Therefore, the following group of transformations are introduced:

$$\begin{aligned} &\psi = \nu_f Gr_x^{1/4}\left\{f(\eta,\xi) + \frac{1}{4}\xi\right\}, \quad \theta = \frac{T-T_\infty}{T_w - T_\infty} \\ &\eta = \frac{y}{x}Gr_x^{1/4}, \qquad \xi = \frac{\rho V_0 x}{\mu_f}Gr_x^{-1/4} \end{aligned} \tag{8}$$

where $\psi$ is the stream function satisfying equation (1) (i.e. $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$). Equations (1)–(3) together with the relations (6) and (7) and the transformations (8) become

$$\left[1+\gamma\left(\theta-\frac{1}{2}\right)\right]f''' + \gamma\theta' f'' + 3ff'' - 2f'^2 + \theta + \xi f'' = \xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) \tag{9}$$

$$\frac{1}{Pr}\left[\left\{1+\frac{4}{3}R_d(1+\Delta\theta)^3\right\}\theta'\right]' + 3f\theta' + \xi\theta' = \xi\left(f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right) \tag{10}$$

where $\gamma$ is the viscosity variation parameter, $\Delta$ $(=\theta_w - 1)$ is the surface temperature variation parameter, $R_d$ the radiation parameter, and $Pr$ the Prandtl number. These parameters are defined below as follows:

$$\theta_w = \frac{T_w}{T_\infty}, \qquad R_d = \frac{4\sigma T_\infty^3}{3k}, \qquad Pr = \left(\frac{\mu C_p}{k}\right)_f \tag{11}$$

Equation (9) is obtained by taking the form of the absolute viscosity as (by Carey and Mollendorf [19]) given below:

$$\mu = \mu_f\left[1+\gamma\left(\theta - \frac{1}{2}\right)\right] \tag{12}$$

The corresponding boundary conditions transform to

$$\begin{aligned} &f(0,\xi) = f'(0,\xi) = 0, \qquad \theta(0,\xi) = 1 \\ &f'(\infty,\xi) = \theta(\infty,\xi) = 0 \end{aligned} \tag{13}$$

Near the leading edge, in the region where the scaled streamwise variable, treated as the local suction parameter, $\xi$, is small, a perturbation solution is expected. Furthermore, an asymptotic solution for large $\xi$ is also possible, which will be obtained later.

However, the solution methodology of equations (9) and (10) with the boundary conditions given in equation (13) for the entire $\xi$ values based on Keller box scheme is proposed here. The scheme specifically incorporated a nodal distribution favoring the vicinity of the plate, enabling accuracy to be maintained in this region of steep gradient. In detail, equations (9) and (10) are solved as a set of five simultaneous equations

$$\frac{\partial f}{\partial \eta} = U, \qquad \frac{\partial U}{\partial \eta} = V, \qquad \frac{\partial \theta}{\partial \eta} = \Theta \tag{14a}$$

$$\frac{\partial}{\partial \eta}\left[1+\gamma\left(\theta-\frac{1}{2}\right)\right]V' + \gamma\Theta V + P_1 fV - P_2U^2 + \theta + \xi U = \xi\left(U\frac{\partial U}{\partial \xi} - V\frac{\partial f}{\partial \xi}\right) \tag{14b}$$

$$\frac{\partial}{\partial \eta}\left[\left\{1+\frac{4}{3}R_d(1+\Delta\theta)^3\right\}\Theta\right] + P_1 fG + \xi\Theta = \xi\left(U\frac{\partial \theta}{\partial \xi} - \Theta\frac{\partial f}{\partial \xi}\right) \tag{14c}$$

where

$$P_1 = 3 \quad \text{and} \quad P_2 = 2 \tag{14d}$$

A net is placed on the $(\eta,\xi)$ plane defined by

$$\begin{aligned} &\xi_0 = 0, \quad \xi_n = \xi_{n-1} + k_n, \qquad n = 1,2,\ldots,N \\ &\eta_0 = 0, \quad \eta_j = \eta_{j-1} + h_j, \qquad j = 1,2,\ldots,J \end{aligned} \tag{15}$$

The range of $\xi$ in $0 \le \xi \le 10$ has been represented by 501 uniform nodal values starting with stem length 0.02. The $\eta_j$ were chosen so that the outer boundary $\eta_J = 25.02$ and were sufficiently dense in the vicinity of the boundary to ensure accuracy when the velocity distribution maximum come close to boundary. In all cases, the nodal points were obtained using the formula $\eta_j = \sinh((j-1)/\Delta\eta)$ with $N = 401$ and $\Delta\eta = 100$.

If $g_j^n$ denotes the value of any variable at $(\xi_n, \eta_j)$, then the variables and derivatives of equations (14b) and (14c) at $(\xi_{n-1/2}, \eta_{j-1/2})$ are replaced by

$$\begin{aligned} g_{j-1/2}^{n-1/2} &= \frac{1}{4}\left(g_j^n + g_{j-1}^n + g_j^{n-1} + g_{j-1}^{n-1}\right) \\ \left(\frac{\partial g}{\partial \xi}\right)_{j-1/2}^{n-1/2} &= \frac{1}{2k_n}\left(g_j^n + g_{j-1}^n - g_j^{n-1} - g_{j-1}^{n-1}\right) \\ \left(\frac{\partial g}{\partial \eta}\right)_{j-1/2}^{n-1/2} &= \frac{1}{2h_n}\left(g_j^n - g_{j-1}^n + g_j^{n-1} - g_{j-1}^{n-1}\right) \end{aligned} \tag{16}$$

Equation (14a), where $\xi$ is not involved explicitly, is approximated at $(\xi_n, \eta_{j-1/2})$ as follows:

$$\begin{aligned} g_{j-1/2}^n &= \frac{1}{2}\left(g_j^n + g_{j-1}^n\right) \\ \left(\frac{\partial g}{\partial \eta}\right)_{j-1/2} &= \frac{1}{h_n}\left(g_j^n - g_{j-1}^n\right) \end{aligned} \tag{17}$$

We now consider the net rectangle on the $(\xi, \eta)$ plane and denote the net points by

$$\begin{aligned} &\xi^0 = 0, \quad \xi^i = \xi^{i-1} + k_i, \qquad i = 1, 2, \ldots, N \\ &\eta_0 = 0, \quad \eta_j = \eta_{j-1} + l_j, \qquad j = 1, 2, \ldots, J, \ \eta_j = \eta_\infty \end{aligned} \tag{18}$$

where $i$ and $j$ are the index points on the $(\xi, \eta)$ plane, and $k_i$ and $l_j$ give the variable mesh width.

The quantities $(f, U, V, \theta, \Theta)$ are approximated at points $(\xi^i, \eta_j)$ of the net by the net function $(f_j^i, U_j^i, V_j^i, \theta_j^i, \Theta_j^i)$. The notation $m_j^i$ is also employed for any net function quantities midway between the net points as follows:

$$\xi^{i-1/2} = \frac{1}{2}\left(\xi^i + \xi^{i-1}\right), \qquad \eta_{j-1/2} = \frac{1}{2}\left(\eta_j + \eta_{j-1}\right) \tag{19a}$$

$$m_j^{i-1/2} = \frac{1}{2}\left(m_j^i + m_j^{i-1}\right), \qquad m_{j-1/2}^i = \frac{1}{2}\left(m_j^i + m_{j-1}^i\right) \tag{19b}$$

The finite-difference approximation of equation (14a) using central finite-difference quotients and average about the mid point $(\xi^i, \eta_{j-1/2})$ can be written as follows:

$$\begin{aligned} \frac{f_j^i - f_{j-1}^i}{h_j} &= U_{j-1/2}^i \\ \frac{U_j^i - U_{j-1}^i}{h_j} &= V_{j-1/2}^i \\ \frac{\theta_j^i - \theta_{j-1}^i}{h_j} &= \Theta_{j-1/2}^i \end{aligned} \tag{20}$$

Similarly, equations (14b) and (14c) can be expressed in finite-difference form, by approximating the functions and their derivatives by central differences about the mid-points $(\xi^{i-1/2}, \eta_{j-1/2})$, giving the following nonlinear difference equations:

$$\begin{aligned} &h_j^{-1}\left(1 - \frac{1}{2}\gamma + \gamma\theta_{j-1/2}^i\right)\left(V_j^i - V_{j-1}^i\right) + \gamma(\Theta V)_{j-1/2}^i \\ &\quad + \alpha_1(f\gamma)_{j-1/2}^i - \alpha_2\left(U^2\right)_{j-1/2}^i + \theta_{j-1/2}^i \\ &\quad + \xi^i V_{j-1/2}^i + \alpha_0\left(V_{j-1/2}^{i-1} f_{j-1/2}^i - f_{j-1/2}^{i-1} V_{j-1/2}^i\right) \\ &= R_{j-1/2}^{i-1} \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{1}{Pr} h_j^{-1}\left\{1 + \frac{4}{3} R_{\mathrm{d}}\left(1 + \Delta\theta_{j-1/2}^i\right)^3\right\}\left(\Theta_j^i - \Theta_{j-1}^i\right) \\ &\quad + \alpha_1(f\Theta)_{j-1/2}^i + \xi^i \Theta_{j-1/2}^i \\ &\quad + \alpha_0\left[U_{j-1/2}^{i-1}\theta_{j-1/2}^i - U_{j-1/2}^i \theta_{j-1/2}^{i-1} + \Theta_{j-1/2}^i f_{j-1/2}^{i-1} \right. \\ &\qquad \left. - \Theta_{j-1/2}^{i-1} f_{j-1/2}^i\right] = T_{j-1/2}^{i-1} \end{aligned} \tag{22}$$

where

$$R_{j-1/2}^{i-1} = -L_{j-1/2}^{i-1} + \alpha_0\left[(fV)_{j-1/2}^{i-1} - \left(U^2\right)_{j-1/2}^{i-1}\right] \tag{23}$$

$$T_{j-1/2}^{i-1} = -M_{j-1/2}^{i-1} + \alpha_0\left[(f\Theta)_{j-1/2}^{i-1} - (U\theta)_{j-1/2}^{i-1}\right] \tag{24}$$

$$\begin{aligned} L_{j-1/2}^{i-1} = \Big[ &h_j^{-1}\left(1 - \frac{1}{2}\gamma + \gamma\theta_{j-1/2}\right)(V_j - V_{j-1}) \\ &+ \gamma(\Theta V)_{j-1/2} + \alpha_1(fV)_{j-1/2} \\ &- \alpha_2\left(U^2\right)_{j-1/2} + \xi_{j-1/2}^i V_{j-1/2} + \theta_{j-1/2}\Big]^{i-1} \end{aligned} \tag{25}$$

$$\begin{aligned} M_{j-1/2}^{i-1} = \Big[ &\frac{1}{Pr} h_j^{-1}\left\{1 + \frac{4}{3} R_{\mathrm{d}}(1 + \Delta\theta_{j-1/2})^3\right\} \\ &\cdot (\Theta_j - \Theta_{j-1}) + \alpha_1(f\Theta)_{j-1/2} \\ &+ \xi_{j-1/2}\Theta_{j-1/2}\Big]^{i-1} \end{aligned} \tag{26}$$

$$\alpha_0 = \frac{\xi^{i-1/2}}{k_j}, \qquad \alpha_1 = P_1 + \alpha_0, \qquad \alpha_2 = P_2 + \alpha_0 \tag{27}$$

The boundary conditions become:

$$f_0^i = 0, \qquad U_0^i = 0, \qquad \theta_0^i = 1$$
$$U_J^i = 0, \qquad \theta_J^i = 0 \tag{28}$$

If we assume $f_j^{i-1}$, $U_j^{i-1}$, $V_j^{i-1}$, $\theta_j^{i-1}$ and $\Theta_j^{i-1}$ to be known for $0 \le j \le J$, equations (21) and (22) are a system of $5J + 5$ equations for the solution of $5J + 5$ unknowns $f_j^i$, $U_j^i$, $V_j^i$, $\theta_j^i$ and $\Theta_j^i$ and $j = 0, 1, 2, \ldots, J$. These nonlinear systems of algebraic equations are linearized by means of Newton's method, which are then solved in a very efficient manner by using the Keller box method, as discussed by Cebeci and Bradshaw [22] in a simpler way. For simplicity of notation we shall write the unknowns at $\xi = \xi^i$ as $(f_j^i, U_j^i, V_j^i, \theta_j^i, \Theta_j^i) \equiv (f_j, U_j, V_j, \theta_j, \Theta_j)$.

Equations (21) and (22) are then linearized by Newton's quasi-linearization technique and then solved using Keller box method taking the initial interaction with given set of converged solution at $\xi = \xi_{n-1}$. To initiate the process with $\xi = \xi_0 = 0$, we first prescribe the set of solution profiles for the functions $(f, U, V, \theta, \Theta)$ from the exact solutions of the set of equations

$$\left[1 + \gamma\left(\theta - \frac{1}{2}\right)\right] f''' + \gamma\theta' f'' + 3ff'' - 2f'^2 + \theta = 0 \tag{29}$$

$$\frac{1}{Pr}\left[\left\{1 + \frac{4}{3} R_{\mathrm{d}}(1 + \Delta\theta)^3\right\}\theta'\right]' + 3f\theta' + \xi\theta' = 0 \tag{30}$$

satisfying the boundary conditions

$$f(0) = f'(0) = 0, \qquad \theta(0) = 1$$
$$f'(\infty) = \theta(\infty) = 0 \tag{31}$$

It should be noted here that, in the absence of the effect of radiation ($R_{\mathrm{d}} = 0$), Carey and Mollendorf [19] have integrated the equations (29) and (30) along with the boundary conditions given by equation (31), using the Nachtsheim–Swigert iteration technique [23] for various values of the Prandtl number ($1 \le Pr \le 1\,000$) and viscosity variation parameter ($-1.6 \le \gamma \le 1.6$). On the other hand, for constant viscosity case (i.e. $\gamma = 0$) solutions of the above equations have been obtained by Hossain et al. [18] for $Pr = 1$, 5, 7 and 10 with the variation of the radiation parameter $R_{\mathrm{d}}$ and the surface temperature $\theta_{\mathrm{w}}$.

## 3. RESULTS AND DISCUSSIONS

Complete detailed solutions have been obtained as described in the preceding section for wide ranged values of the physical parameters $\gamma$, $R_{\mathrm{d}}$ and $\theta_{\mathrm{w}}$ against $\xi$ in $[0, 10]$ although in some representations its upper limit has been shown to be 5.

The quantities of physical significance are the velocity and the temperature fields as well as the skin friction and heat transfer coefficients. Solutions of equations (9) and (10) enables us to find the nondimensional velocity components $\bar{u}$ and $\bar{v}$ from the following expressions:

$$\bar{u} = \frac{V}{g\beta(T_{\mathrm{w}} - T_\infty)}\, u = \xi^2 f'(\eta, \xi) \tag{32}$$

$$\bar{v} = \frac{v}{V} = \xi^{-1}\left\{3f - \eta' + \xi + \xi\,\frac{\partial f}{\partial \xi}\right\} \tag{33}$$

The effect of the transpiration velocity variations on the velocity and temperature profiles for different values of the viscosity variation parameter is shown in *figures 2* and *3*, respectively. This evaluation is illustrated for sample values of $\xi = 0.0$, 0.5, 1, 2 and 3 for fluid having $Pr = 1.0$ while the radiation parameter $R_{\mathrm{d}} = 1.0$ and the surface temperature $\theta_{\mathrm{w}} = 1.5$ for $\gamma = -1.0$, 0.0 and 1.0, respectively. As the local suction parameter increases, the maximum fluid velocity decreases. This is due to the fact that the effect of the suction is to take away the warm fluid on the vertical plate and thereby decreasing the maximum velocity with a decreasing in the intensity of the natural convection rate as depicted by *figure 2*. Moreover, it is worth noting that the limiting asymptotic state approaches at larger values of the local suction parameter $\xi$. *Figure 3* shows the effect of the local suction parameter on the temperature profiles for various values of the viscosity variation parameter. It is interesting to note that the temperature profiles decrease with an increase in the suction effect. As the suction rate is increased, more warm fluid is taken away and, thus, the thermal boundary layer thickness decreases.

The quantities of physical interested, namely, the local skin friction, $C_{\mathrm{fx}}$, and the rate of heat transfer in terms of local Nusselt number, $Nu_{\mathrm{x}}$, are prescribed by

$$\frac{C_{\mathrm{fx}} Gr_{\mathrm{x}}^{-3/4}}{1 + \gamma/2} = f''(0, \xi) \tag{34}$$

$$\frac{Nu_{\mathrm{x}} Gr_{\mathrm{x}}^{-1/4}}{1 + 4R_{\mathrm{d}}\theta_{\mathrm{w}}^3/3} = -\theta'(0, \xi) \tag{35}$$

Precise plots of the local skin friction coefficient and the local Nusselt number are presented in *figure 4* for different values of the viscosity variation parameter and the radiation parameter for Prandtl number $Pr = 1$. In this figure, the dotted and the dashed lines correspond to the perturbation solutions obtained for small and

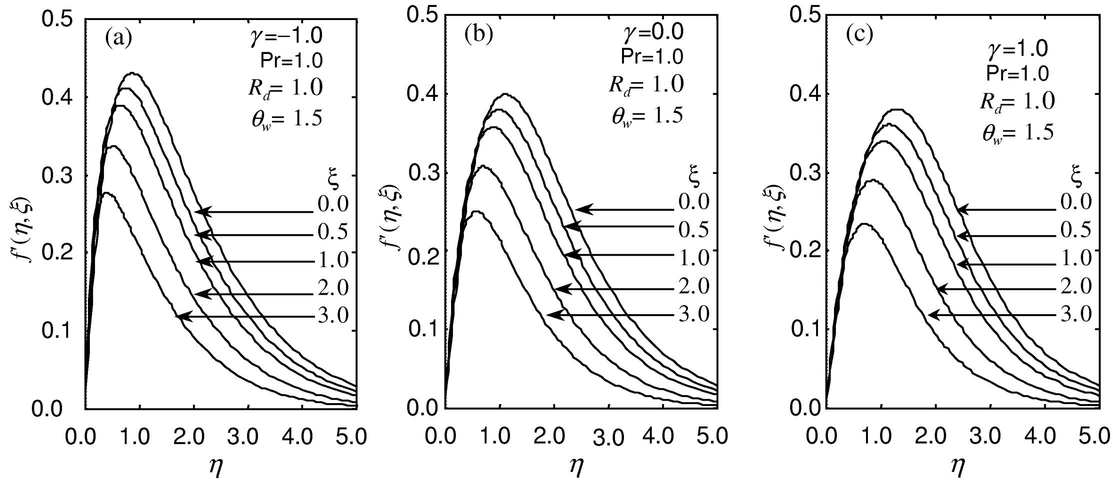


**Figure 2.** Velocity profiles for different $\xi$ against $\eta$: (a) $\gamma = -0.5$, (b) $\gamma = 0.0$ and (c) $\gamma = 0.5$ while $Pr = 1.0$, $R_\mathrm{d} = 1.0$ and $\theta_\mathrm{w} = 1.5$. The profiles for $\gamma = 0.0$ are those obtained by Hossain et al. [13].

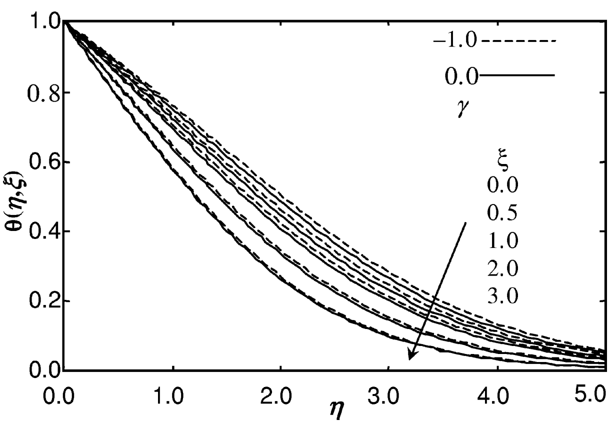


**Figure 3.** Numerical values of temperature function against $\eta$ for different $\xi$ while $Pr = 1.0$, $R_\mathrm{d} = 1.0$ and $\theta_\mathrm{w} = 1.5$: $\gamma = -1.0$ and 0.0.

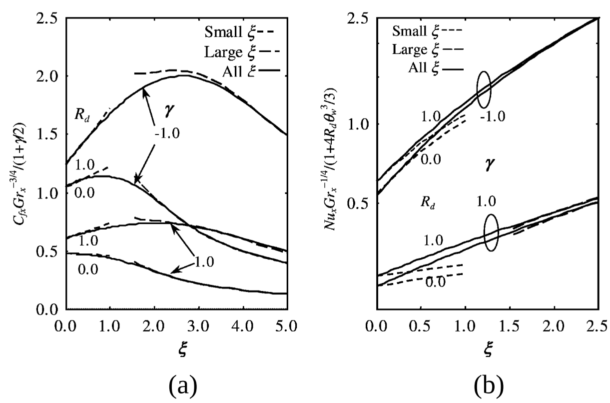


**Figure 4.** Numerical values of (a) shear-streass coefficient and (b) heat transfer coefficient against $\xi$ for $R_\mathrm{d} = 0.0$ and 1.0, $\gamma = -1.0$ and 1.0 while $Pr = 1.0$ and $\theta_\mathrm{w} = 1.5$.

large values of $\xi$, respectively. For two values of the viscosity variation parameter, i.e. $\gamma = -1.0$ and 1.0 with and without the effect of radiation, the numerical values of the local skin friction and the local Nusselt number coefficients are depicted in *figures 4(a)* and *4(b)*, respectively. *Figure 4(a)* shows that the numerical values of the local skin friction coefficient $f''(0,\xi)$ diminish for larger values of $\xi$ $(= 10)$. As mentioned before, the maximum velocity is decreased with an increase in the suction rate and thereby decreases the wall shear stress and the hydrodynamic boundary layer thickness. For strong suction rate, the local skin friction coefficient approaches zero as shown in this figure. Consequently, the local heat transfer coefficient is increased with an increase in the suction rate due to the fact that the thermal boundary layer thickness decreases with an increase in the suction rate as shown in *figure 4(b)*. The effect of the radiation on the local skin friction and the heat transfer coefficients is shown in *figures 4(a)* and *4(b)*, respectively, for various viscosity variation parameters. It can be seen from these figures that an increase in the value of the radiation parameter $R_\mathrm{d}$ leads to a significant change in the values of the local skin friction coefficient and slight change in the local heat transfer coefficient. This can be attributed to the thermal radiation interaction enhancement, which increases the fluid velocity and, consequently, increases the velocity gradient at the wall. For fixed value of the viscosity variation parameter, *figure 4(a)* shows that the peak value of the local skin friction coefficient increases with an increase in the radiation conduction parameter and its location moves away from the leading edge. This situation is reversed for positive value of $\gamma$ where the peak value occurs at the leading edge.

The effect of the viscosity variation parameter with and without the presence of the radiation on the local heat transfer rate is shown in *figure 4(b)*. It can be seen from this figure that the viscosity variation parameter has a significant effect on the local Nusselt number. As the viscosity variation parameter decreases, the thermal boundary layer thickness decreases and, thus, the rate of the heat transfer increases. The viscosity variation parameter also has a noticeable effect on the local skin friction coefficient as shown in *figure 4(a)*. Decreasing the viscosity within the boundary layer leads to an increase in the velocity within the layer and, thus, increases the local skin friction coefficient.

## 4. COMPARISON WITH PERTURBATION SOLUTIONS

It should be intimated that the problem under consideration is susceptible to perturbation analysis. Indeed, earlier work on this problem for fluid with uniform viscosity focused on this approach. It is of interest then to assess the value of approximate series representations in forecasting the essential physical parameters of the problem, namely, the local skin friction coefficient and the local Nusselt number. In particular, we are interested to compare with exact numerical solutions to investigate the variation with $\gamma$ of indeterminacy of the downstream series solution.

### 4.1. Small $\xi$

The series solution is valid for sufficiently small values of $\xi$, that is, either for sufficiently small distances from the leading edge or for small values of $V$ (see the definition of $\xi$ in (8)). Accordingly, the functions $f(\eta,\xi)$ and $\theta(\eta,\xi)$ are expanded in a power series in $\xi$ as follows:

$$f(\eta,\xi) = f_0(\eta) + \xi f_1(\eta) + \cdots \tag{36}$$

and

$$\theta(\eta,\xi) = \theta_0(\eta) + \xi\theta_1(\eta) + \cdots \tag{37}$$

Substituting the above expansion into equations (9) and (10) and equating the coefficients of various powers of $\xi$, we get the following equations:

$$(1+\gamma\theta_0) f_0''' + \gamma\theta_0' f_0'' + 3 f_0 f_1'' - 2 f_0'^2 + \theta_0 = 0 \tag{38a}$$

$$\frac{1}{Pr}\left[1 + \frac{4}{3} R_d (1+\Delta\theta_0)^3\right]\theta_0'' + \frac{4}{Pr}\Delta R_d (1+\Delta\theta_0)^2 \theta_0'^2 + 3 f_0\theta_0' = 0 \tag{38b}$$

$$f_0(0) = f_0'(0) = 0, \qquad \theta_0(0) = 1, \qquad f_0'(\infty) = \theta_0(\infty) = 0 \tag{38c}$$

$$(1+\gamma\theta_0) f_1''' + \gamma\left(\theta_0' f_1'' + f_0''\theta_1' + f_0'''\theta_1'\right) + 3 f_0 f_1'' + 4 f_0' f_1 - 3 f_0' f_1' + f_0'' + \theta_1 = 0 \tag{39a}$$

$$\frac{1}{Pr}\left[1 + \frac{4}{3} R_d (1+\Delta\theta_0)^3\right]\theta_1'' + \frac{4}{Pr}\Delta R_d (1+\Delta\theta_0)^2\left(2\theta_0'\theta_1' + \theta_0''\theta_1\right) + \frac{8}{Pr}\Delta^2 R_d (1+\Delta\theta_0)^2\theta_0'^2\theta_1 + 3 f_0\theta_1' + 4\theta_0' f_1 - f_0'\theta_1 + \theta_0 = 0 \tag{39b}$$

$$f_1(0) = f_1'(0) = 0, \qquad \theta_1(0) = 0, \qquad f_1'(\infty) = \theta_1(\infty) = 0 \tag{39c}$$

In *tables I* and *II*, the boundary values data for different $\gamma$ and $R_d$ while $Pr = 1$ and $\theta_w = 1.5$ are already indicated for associated nonlinear equations governing $(f_0, \theta_0)$. To save space, solutions of the linear equations associated to $(f_1, \theta_1)$ are not shown. Knowing the solutions of these equations, the values of the skin friction and heat transfer coefficient are then approximated from the following relations:

$$\frac{C_{fx} Gr_x^{-3/4}}{1+\gamma/2} = \left(f_0''(Pr,\gamma,R_d) + \xi f_1''(Pr,\gamma,R_d,\theta_w) + \cdots\right)_{\eta=0} \tag{40}$$

and

$$\frac{Nu_x Gr_x^{-1/4}}{1 + 4R_d\theta_w^3/3} = \left(\theta_0'(Pr,\gamma,R_d,\theta_w) + \xi''\theta_1'(Pr,\gamma,R_d,\theta_w) + \cdots\right)_{\eta=0} \tag{41}$$

*Figure 4* indicates the departure of the series representations from the exact numerical solutions represented in terms of the local skin friction and the local Nusselt number.

### 4.2. Large $\xi$

Now attention shall be given to the behavior of the solution to equations (9) and (10) when $\xi$ is large. Given that we consider only the suction case, for which we take the positively signed terms in equations (9) and (10), an order-of-magnitude analysis of the various terms in these equations shows that the largest terms are $\xi f'''$ in equation (9) and $\xi\theta'$ in equation (10). In their respective equations, both terms have to be balanced and the only way to do this is to assume that $\eta$ is small and, hence, $\eta$-derivatives are large. Given that $\theta = \mathrm{O}(1)$ as $\xi \to \infty$, it is necessary to find the appropriate scaling for $f$ and $\eta$. On balancing the $f'''$, $\theta$ and $\xi f''$ terms in equation (9), it is found that $\eta = \mathrm{O}(\xi^{-1})$ and $f = \mathrm{O}(\xi^{-3})$ as $\xi \to \infty$. Therefore, the following substitutions are made:

$$f = \xi^{-3} F(\zeta, \xi), \qquad \theta = G(\zeta, \xi), \qquad \zeta = \xi\eta \quad (42)$$

Equations (9) and (10) then take the following form:

$$\left[1 + \gamma\left(G - \frac{1}{2}\right)F''\right]' + F'' + G = \xi^{-3}\left(F'\frac{\partial F'}{\partial \xi} - F''\frac{\partial F}{\partial \xi}\right) \quad (43)$$

$$\left[\left\{1 + \frac{4}{3}R_{\mathrm{d}}(1+\Delta G)^3\right\}G'\right]' Pr + G' = Pr\,\xi^{-3}\left(F'\frac{\partial G}{\partial \xi} - G'\frac{\partial F}{\partial \xi}\right) \quad (44)$$

where the primes denote differentiation with respect to $\zeta$. This convention will be used for the rest of this section.

The boundary conditions can be written as follows:

$$F(0,\xi) = F'(0,\xi) = 0, \qquad G(0,\xi) = 0$$
$$F'(\infty,\xi) = G(\infty,\xi) = 0 \quad (45)$$

For extremely large $\xi$ the right-hand side of equations (43) and (44) can be easily neglected to get

$$(1+\gamma G)F''' + \gamma G'F'' + F'' + G = 0 \quad (46)$$

$$\left\{1 + \frac{4}{3}R_{\mathrm{d}}(1+\Delta G)^3\right\}G'' + \frac{4}{Pr}\Delta R_{\mathrm{d}}(1+\Delta G)^2 G'^2 + PrG' = 0 \quad (47)$$

$$F(0) = F'(0) = 0, \qquad G(0) = 0$$
$$F'(\infty) = G(\infty) = 0 \quad (48)$$

Retaining all the pertinent parameters, closed form solutions of the above equations are not possible. The

TABLE I
Calculated values of $f''(0)$ and $\theta'(0)$ from equations (29) and (30) for different $\gamma$ and $R_{\mathrm{d}}$ ($Pr = 1.0$ and $\theta_{\mathrm{w}} = 1.5$).

| $\gamma$ | −1.6 | 0.0 | 0.8 | 1.6 | −1.6 | 0.0 | 0.8 | 1.6 |
|---|---|---|---|---|---|---|---|---|
| $R_{\mathrm{d}}$ | $f''(0)$ | | | | $\theta'(0)$ | | | |
| 0.0* | 2.0416 | 0.6422 | 0.5050 | 0.4233 | 0.6514 | 0.5671 | 0.5469 | 0.5315 |
| 0.0 | 2.0411 | 0.6421 | 0.5050 | 0.4222 | 0.6514 | 0.5671 | 0.5469 | 0.5281 |
| 0.5 | 2.1934 | 0.7502 | 0.5994 | 0.5068 | 0.3604 | 0.3209 | 0.3107 | 0.3026 |
| 1.0 | 2.2579 | 0.7988 | 0.6423 | 0.5451 | 0.2808 | 0.2538 | 0.2466 | 0.2408 |
| 2.0 | 2.3158 | 0.8498 | 0.6886 | 0.5864 | 0.2115 | 0.1951 | 0.1900 | 0.1865 |
| 3.0 | 2.3421 | 0.8776 | 0.7150 | 0.6095 | 0.1773 | 0.1660 | 0.1615 | 0.1596 |

* Values in this row are due to Carey and Mollendorf [19].

TABLE II
Calculated values of $f''(0)$ and $\theta'(0)$ from equations (29) and (30) for different $\gamma$ and $\theta_{\mathrm{w}}$ ($Pr = 1.0$ and $R_{\mathrm{d}} = 1.0$).

| $\gamma$ | −1.6 | 0.0 | 0.8 | 1.6 | −1.6 | 0.0 | 0.8 | 1.6 |
|---|---|---|---|---|---|---|---|---|
| $\theta_{\mathrm{w}}$ | $f''(0)$ | | | | $\theta'(0)$ | | | |
| 0.01 | 2.2001 | 0.7295 | 0.5781 | 0.4877 | 0.4624 | 0.4120 | 0.3997 | 0.3890 |
| 0.1 | 2.2109 | 0.7425 | 0.5899 | 0.4984 | 0.4212 | 0.3761 | 0.3650 | 0.3553 |
| 0.5 | 2.2579 | 0.7988 | 0.6409 | 0.5451 | 0.2808 | 0.2538 | 0.2470 | 0.2408 |
| 1.0 | 2.3064 | 0.8579 | 0.6944 | 0.5944 | 0.1819 | 0.1675 | 0.1639 | 0.1600 |
| 1.5 | 2.3382 | 0.9025 | 0.7348 | 0.6326 | 0.1277 | 0.1196 | 0.1182 | 0.1151 |

asymptotic skin friction coefficient and local Nusselt number are calculated from the following relations:

$$\frac{C_{\mathrm{fx}}Gr_{\mathrm{x}}^{-3/4}}{1+\gamma/2} \approx \xi^{-1}F''(\gamma, R_{\mathrm{d}}, \theta_{\mathrm{w}}, \xi)_{\eta=0} \tag{49}$$

$$\frac{Nu_{\mathrm{x}}Gr_{\mathrm{x}}^{-1/4}}{1+4R_{\mathrm{d}}\theta_{\mathrm{w}}^{3}/3} \approx \xi G'(\gamma, R_{\mathrm{d}}, \theta_{\mathrm{w}}, \xi)_{\eta=0} \tag{50}$$

Numerical values of the local skin friction coefficient and the local Nusselt number obtained from the relations given in equations (49) and (50) are depicted in *figure 4* for different values of $\gamma$ and $R_{\mathrm{d}}$ for $\theta_{\mathrm{w}} = 1.5$ and $Pr = 1$ by the dashed line for comparison with the numerical solutions of equations (9) and (10). The comparison shows excellent agreement between these two solutions which claims that the present solutions can be accepted for experimental verifications by the experimentalist.

## 5. CONCLUSIONS

The present investigation focuses on the effect of thermal radiation on the natural convection flow with variable viscosity from a porous vertical plate. Rosseland diffusion approximation was invoked in this study. At any given values of the radiation parameter and the surface temperature, the local Nusselt number increases due to an increase in the local suction parameter. Moreover, the radiation parameter and the viscosity variation parameter were found to have a significant effect on the local skin friction coefficient and the local Nusselt number for different local suction parameter.

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