REFERENCES

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation." *IEEE Trans. Antennas Propagat.*, vol. AP-34, no. 3, pp. 276–280, Mar. 1986.

[2] S. Haykin *et al.*, "Some aspects of array processing," *Proc. Inst. Elec. Eng.*, pt. F, vol. 139, no. 1, pp. 1–26, Feb. 1992.

[3] T. Shan *et al.*, "On spatial smoothing for direction-of-arrival estimation of coherent signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, no. 4, pp. 806–811, Aug. 1985.

[4] R. T. Williams *et al.*, "An improved spatial smoothing technique for bearing estimation in a multipath environment," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, no. 4, pp. 425–431, Apr. 1988.

[5] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 5, pp. 720–741, May 1989.

[6] B. D. Rao and K. V. S. Hari, "Effect of spatial smoothing on the performance of music and the minimum norm method," *Proc. Inst. Elec. Eng.* , pt. F, vol. 137, no. 6, pp. 449–458, Dec. 1990.

[7] T. Shan *et al.*, "On smoothed rank profile tests in eigenstructure methods for directions-of-arrival estimation." *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, no. 10, pp. 1377–1385, Oct. 1987.

[8] J. S. Thompson, P. M. Grant, and B. Mulgrew, "Correspondence: generalized algorithm for DOA estimation in a passive sonar," *Proc. Inst. Elec. Eng.*, pt. F, vol. 140, no. 5, pp. 339–340, Oct. 1993.

[9] S. U. Pillai and H. K. Byung, "Forward/backward spatial smoothing techniques for coherent signal identification," *IEEE Trans., Acoust., Speech, Signal Processing*, vol 37, no. 1, pp. 8–14, Jan. 1989.

[10] M. Kaveh and A. J. Barabell, "The statistical performance of the music and the minimum-norm algorithms in resolving plane waves in noise," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, no. 2, pp. 331–340, Apr. 1986.

[11] S. U. Pillai and H. K. Byung, "Performance analysis of music-type high-resolution estimators for direction finding in correlated and coherent sources." *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 8, pp. 1176–1189, Aug. 1989.

[12] L. Chang and C. Yeh, "Resolution threshold for coherent sources using smoothed eigenstructure methods," *Proc. Inst. Elec. Eng.*, pt. F, vol. 138, no. 5, pp. 470–478, Oct. 1991.

[13] B. Porat and B. Friedlander, "Analysis of the asymptotic relative efficiency of the music algorithm," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, no. 4, pp. 532–543, Apr. 1988.

[14] A. J. Weiss and B. Friedlander, "Performance analysis of spatial smoothing with interpolated arrays," *IEEE Trans. Signal Processing*, vol. 41, no. 5, pp. 1881–92, May 1993.

[15] D. H. Brandwood, "A complex gradient operator and its application in adaptive array theory," *Proc. Inst. Elec. Eng.*, pt. F, vol. 130, no. 1, pp. 11–16, Feb 1983.

[16] V. U. Reddy *et al.*, "Performance analysis of the optimum beamformer in the presence of correlated sources and its behavior under spatial smoothing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, no. 7, pp. 927–936, July 1987.

On Rank of Block Hankel Matrix for 2-D Frequency Detection and Estimation

Howard Hua Yang and Yingbo Hua

Abstract—**For detection and estimation of 2-D frequencies from a 2-D array of data using a subspace decomposition method, one needs to construct a block Hankel matrix. For reliable detection and estimation, the rank of the block Hankel matrix should be made equal to the number of 2-D frequencies inherent in the data in the absence of noise. In this work, we provide the conditions for achieving the desired rank.**

I. INTRODUCTION

It is well known that for detection and estimation of 1-D frequencies from a single sequence of data using a subspace decomposition method, one needs to form a Hankel matrix,(e.g., see [4]). The idea behind the Hankel matrix is also equivalent to the so-called moving-window-average in the context of array processing where multiple data sequences are available, (e.g., see [3]). Similarly, for detection and estimation of 2-D frequencies from a single 2-D array of data using a subspace decomposition method, one needs to form a block Hankel matrix where each block is a Hankel matrix. The block Hankel matrix is referred to as enhanced data matrix in [2]. An important property of the block Hankel matrix is that its rank can be equal to the number of 2-D frequencies inherent in the data in the absence of noise regardless of the distribution of the 2-D frequencies. This property clearly suggests that the number of the 2-D frequencies can be detected using the singular values of the block Hankel matrix. Furthermore, as shown in [2], if the rank of the block Hankel matrix is equal to the desired number (number of 2-D frequencies) in the absence of noise, the 2-D frequencies can be efficiently estimated from the principal subspace of the block Hankel matrix using a matrix pencil approach. A sufficient condition for the block Hankel matrix to achieve the desired rank was given in [2]. In this work, we provide a more complete and more rigorous analysis of the block Hankel matrix and show a number of possible (sufficient or/and necessary) conditions under which the block Hankel matrix has the desired rank.

In Section II, the block Hankel matrix is defined and decomposed in structure for later analysis. In Section III, the sufficient or/and necessary conditions for the rank property are discussed in detail. The main contribution of this paper is illustrated in Fig. 1.

II. THE BLOCK HANKEL MATRIX

Consider a 2-D array of data defined by

$$x(m, n) = \sum_{i=1}^{N_s} a_i y_i^m z_i^n, \quad m = 0, 1, \cdots, M-1, \quad n = 0, 1, \cdots, N-1 \tag{1}$$

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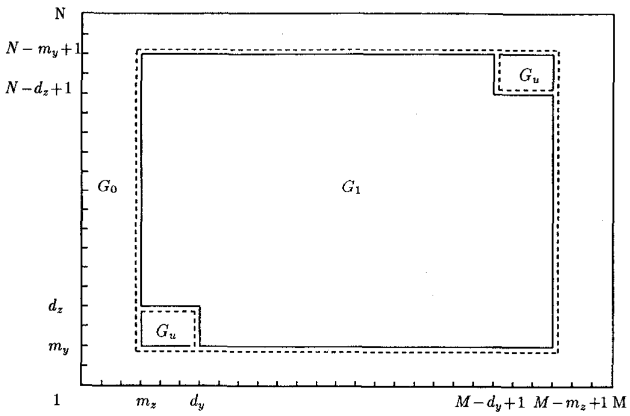

Fig. 1. Regions of window size parameters. For region G_1, the block Hankel matrix is of the desired rank (the number of 2-D sinusoids). For region G_0, the block Hankel matrix is of a rank less than the desired. Region G_u is the uncertain region where the block Hankel matrix may or may not have its rank equal to the desired. M and N define the size of the original data set. m_y and m_z denote the maximum multiplicities of poles in the first and second dimensions, respectively. d_y and d_z denote the numbers of distinct poles in the first and second dimensions, respectively. The sufficient condition given in [2] is a subset of G_1.

where a_i is the ith complex amplitude (none zero); $(y_i, z_i) = (e^{j\omega_{1i}}, e^{j\omega_{2i}})$ defines the ith 2-D frequency $(\omega_{1i}, \omega_{2i})$; N_s is the number of the 2-D frequencies. Naturally, $\{(y_i, z_i)\}$ should be distinct. Let d_y and d_z be the numbers of distinct poles in $\{y_i\}$ and $\{z_i\}$, respectively. Let m_y and m_z be the maximum multiplicity in $\{y_i\}$ and the maximum multiplicity in $\{z_i\}$, respectively.

It follows from the definition that $m_z \leq d_y$ and $m_y \leq d_z$.

The block Hankel matrix of $x(m, n)$ is defined as

$$X_e = \begin{bmatrix} X_0 & X_1 & \cdots & X_{M-K} \\ X_1 & X_2 & \cdots & X_{M-K+1} \\ \cdots & \cdots & \cdots & \cdots \\ X_{K-1} & X_K & \cdots & X_{M-1} \end{bmatrix}$$

where for $m = 0, 1, \cdots, M-1$,

$$X_m = \begin{bmatrix} x(m,0) & x(m,1) & \cdots & x(m, N-L) \\ x(m,1) & x(m,2) & \cdots & x(m, N-L+1) \\ \cdots & \cdots & \cdots & \cdots \\ x(m, L-1) & x(m, L) & \cdots & x(m, N-1) \end{bmatrix}$$

in which K and L are called the window-size parameters.

Based on (1), the block Hankel matrix X_e can be decomposed as follows. We define

$$V(v, r) \hat{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_{N_s} \\ \cdots & \cdots & \cdots & \cdots \\ v_1^{r-1} & v_2^{r-1} & \cdots & v_{N_s}^{r-1} \end{pmatrix}$$

where $v = (v_1, v_2, \cdots, v_{N_s})^T$ is a vector. Then we can write (also shown in [2])

$$\begin{aligned} X_e &\hat{=} X_e(K, L) \\ &= E_l A_d E_r \end{aligned} \tag{2}$$

where

$$\begin{aligned} E_l &\hat{=} E_l(K, L) \\ &\hat{=} \begin{bmatrix} V(z, L) \\ V(z, L) Y_d \\ \vdots \\ V(z, L) Y_d^{K-1} \end{bmatrix}_{KL \times N_s} \\ z &\hat{=} (z_1, \cdots, z_{N_s})^T, \\ Y_d &\hat{=} \mathrm{diag}(y_1, \cdots, y_{N_s}), \\ E_r &\hat{=} E_r(K, L) \\ &= E_l^T(M-K+1, N-L+1) \end{aligned}$$

and

$$A_d \hat{=} \mathrm{diag}(a_1, \cdots, a_{N_s}).$$

It is clear from (2) that $\mathrm{rank}(X_e) \leq N_s$. What we need to address next is the conditions that should be satisfied by the window-size parameters K and L such that $\mathrm{rank}(X_e)$ is equal to the desired value N_s. The general results are shown in Fig. 1, and the detailed derivations are given in the next section.

III. The Conditions on the Window Size

Note that the matrix E_l takes a form identical to that of the following matrix:

$$O_k \hat{=} O_k(C, A) \hat{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \tag{3}$$

where $k \geq 1$, and A and C are $n \times n$ and $m \times n$ matrices. When $k = n$, O_n is known as the observability matrix of the pair (C, A). Associated with the matrix O_k, the following two lemmas will be useful.

Lemma 1: Let n_1 be the order of the minimum polynomial of A. Then for $n' \geq n_1$, $\mathrm{rank}(O_{n'}) = n$ iff $\mathrm{rank}([\begin{smallmatrix}\lambda I - A \\ C\end{smallmatrix}]) = n$ for all eigenvalues λ of A.

Proof: This lemma is a generalization of the Theorem 9 in [1] (p. 240). Q.E.D.

Lemma 2: If $\mathrm{rank}([\begin{smallmatrix}\lambda I - A \\ C\end{smallmatrix}]) < n$ for some eigenvalue of A, then $\mathrm{rank}(O_k) < n$ for all $k \geq 1$.

Proof: Let λ be the eigenvalue of A such that $\mathrm{rank}([\begin{smallmatrix}\lambda I - A \\ C\end{smallmatrix}]) < n$. Then there is a vector $x \neq 0$ such that $Ax = \lambda x$ and $Cx = 0$. So $CA^k x = 0$ for all $k \geq 1$. Hence, $O_k x = 0$ for all $k \geq 1$. Therefore, $\mathrm{rank}(O_k) < n$ for all $k \geq 1$. Q.E.D.

Now we are ready to provide the following.

Theorem 1:

a) If $K \geq d_y$, then $\mathrm{rank}[E_l(K, L)] = N_s$ iff $L \geq m_y$;
   if $L \geq d_z$, then $\mathrm{rank}[E_l(K, L)] = N_s$ iff $K \geq m_z$.

b) If $K \leq M - d_y + 1$, then $\mathrm{rank}[E_r(K, L)] = N_s$ iff $L \leq N - m_y + 1$;
   if $L \leq N - d_z + 1$, then $\mathrm{rank}[E_r(K, L)] = N_s$ iff $K \leq M - m_z + 1$.

c) $\mathrm{rank}[X_e(K, L)] = N_s$ if the conditions in a) and b) are met.

Proof: First, we consider $E_l(K, L)$. Let $(C, A) = [V(z, L), Y_d]$. Since A is a diagonal matrix, all eigenvalues of A are the elements on the diagonal. Let $\lambda = y_i$ be an element in $\{y_k\}$ with a multiplicity m, then we can transform, by a simple column permutation as follows:

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \text{ into } \begin{bmatrix} D_1 & 0 \\ 0 & 0 \\ V_1 & V_2 \end{bmatrix} \tag{4}$$

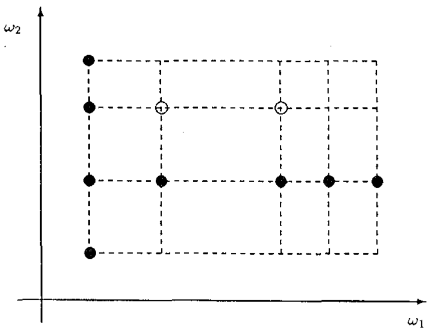

Fig. 2. Constellation of 2-D frequencies that satisfy the condition of Corollary 1.

and, hence, we have

$$\operatorname{rank}\left(\begin{bmatrix}\lambda I - A\\ C\end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix}D_1 & 0\\ 0 & 0\\ V_1 & V_2\end{bmatrix}\right) = \operatorname{rank}(D_1) + \operatorname{rank}(V_2)$$

where $D_1 = \operatorname{diag}(y_i - y_1', \cdots, y_i - y_{N_s-m}')$, $V_2 = V[(z_1', \cdots, z_m')^T, L]$, none of $\{y_1', \cdots, y_{N_s-m}'\} \subset \{y_1, \cdots, y_{N_s}\}$ is equal to y_i, and the poles in $\{z_1', \cdots, z_m'\} \subset \{z_1, \cdots, z_{N_s}\}$ are distinct. So the rank of D_1 is $N_s - m$. The rank of V_2 is m iff $L \geq m$. Hence, $\operatorname{rank}([{}^{\lambda I - A}_{C}]) = N_s$ for $\lambda = y_i$ iff $L \geq m$. Therefore, $\operatorname{rank}([{}^{\lambda I - A}_{C}]) = N_s$ for all $\lambda \in \{y_i\}$ iff $L \geq m_y$. Since A has d_y distinct diagonal elements, the order of the minimum polynomial of A is equal to d_y. So, by Lemma 1, for $K \geq d_y$, $\operatorname{rank}[E_l(K, L)] = N_s$ iff $L \geq m_y$.

We can symmetrically get the other half conclusion of (a) by using the fact that there exists a permutation matrix P [2] such that

$$PE_l = \begin{bmatrix} V(y, K)\\ V(y, K)Z_d\\ \vdots\\ V(y, K)Z_d^{L-1}\end{bmatrix} = O_L[V(y, K), Z_d] \tag{5}$$

where $y \hat{=} (y_1, \cdots, y_{N_s})^T$ and $Z_d = \operatorname{diag}(z_1, \cdots, z_{N_s})$.

Part (b) follows readily from part (a) since $E_r = E_l^T(M - K + 1, N - L + 1)$.

From (2), we know that $\operatorname{rank}(X_e) = N_s$ iff $\operatorname{rank}(E_l) = \operatorname{rank}(E_r) = N_s$. Combining (a) and (b), we have (c). Q.E.D.

Theorem 2:

1) $\operatorname{rank}[E_l(K, L)] < N_s$ if $L < m_y$ or $K < m_z$.
2) $\operatorname{rank}[E_r(K, L)] < N_s$ if $L > N - m_y + 1$ or $K > M - m_z + 1$.
3) $\operatorname{rank}[X_e(K, L)] < N_s$ if $L < m_y$ or $K < m_z$ or $L > N - m_y + 1$ or $K > M - m_z + 1$.

Proof: Following the first paragraph of the proof for Theorem 1, one can show that if $\lambda = y_i$ that has the maximum multiplicity m_y, then $\operatorname{rank}(V_2) < m_y$ if $L < m_y$ and hence $\operatorname{rank}([{}^{\lambda I - A}_{C}]) < N_s$ if $L < m_y$. Therefore, by Lemma 2, $\operatorname{rank}[E_l(K, L)] < N_s$ if $L < m_y$.

Other proofs can be done similarly. Q.E.D.

By Theorems 1 and 2, the whole window size set $G = \{(K, L)\colon 1 \leq K \leq M, 1 \leq L \leq N\}$ is divided into three sets $G = G_0 \cup G_1 \cup G_u$ (see Fig. 1). For the sets G_0 and G_1, we have definite answers: i.e., $\operatorname{rank}[X_e(K, L)] < N_s$ when $(K, L) \in G_0$, and $\operatorname{rank}[X_e(K, L)] = N_s$ when $(K, L) \in G_1$. But for G_u, the answer is uncertain depending on the distribution of the signal poles. In other words, $\operatorname{rank}[X_e(K, L)]$ may be less than or equal to N_s for $(K, L) \in G_u$. It should be noted that the sufficient condition given in [2], i.e., $N_s \leq K \leq M - N_s + 1$ and $N_s \leq L \leq N - N_s + 1$, is a subset of G_1.

When $m_y = d_z$ and $m_z = d_y$, the set G_u is empty. From Theorems 1 and 2, we have the following sufficient and necessary constraint on the window size under which the rank of the enhanced data matrix is equal to the desired number.

Corollary 1: If $m_y = d_z$ and $m_z = d_y$, then

$$\operatorname{rank}[X_e(K, L)] = N_s \text{ iff } d_y \leq K \leq M - d_y + 1,\quad d_z \leq L \leq N - d_z + 1.$$

Note that the condition of Corollary 1 is satisfied when all 2-D frequencies are scattered on a rectangular grid (not necessarily uniform) and at least one straight line in each dimension on the grid is fully occupied by 2-D frequencies, which is illustrated in Fig. 2.

References

[1] F. M. Callier and C. A. Desoer, *Linear System Theory.* New York: Springer-Verlag, 1991.

[2] Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil," *IEEE Trans. Signal Processing,* vol. 40, no. 9, pp. 2267–2280, Sept. 1992.

[3] T. J. Shan *et al.*, "On spatial smoothing for direction-of-arrival estimation of coherent signals," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-33, no. 4, pp. 806–811, Aug. 1985.

[4] D. W. Tufts and R. Kumaresan, "Estimation of frequencies of multiple sinusoids: Making linear prediction perform like maximum likelihood," *Proc. IEEE,* vol. 70, no. 9, pp. 975–989, Sept. 1982.

High-Speed Systolic Ladder Structures for Multidimensional Recursive Digital Filters

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Abstract—**We propose a multilevel approach for designing high-speed systolic ladder structures for multidimensional (MD) recursive digital filters. Based on appropriately selected 1-D filter structures for each filter dimension (or level), a large variety of MD systolic filter structures may be derived. In particular, we introduce a new 1-D filter structure that proves the most suitable structure in terms of a systolic ladder implementation, because it leads to MD ladder filter structures possessing such important properties as the shortest critical path (for filters without order augmentation), the canonic number of high-level storage registers (e.g., line and frame registers of images), and local interconnectivity.**

I. Introduction

High-speed multidimensional (MD) digital filtering is very useful for real-time video signal processing such as video image coding, bandwidth compression, sampling rate conversion and the enhancement of television signals. In this contribution, we are concerned

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