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Strict Identifiability of Multiple FIR Channels Driven by an Unknown Arbitrary Sequence

Yingbo Hua and Mati Wax

***Abstract*—A system of multiple finite impulse response (FIR) channels driven by an unknown input is said to be strictly identifiable if, in the absence of noise, the given channel outputs can only be realized by a unique (up to a constant) system impulse response and a unique input sequence. In this correspondence, we show necessary and sufficient conditions for strict identifiability, and establish a connection among strict identifiability, a cross-relation-based (CR-based) identifiability and a Fisher information-based (FI-based) identifiability.**

I. Introduction

The problem of blind identification of multiple FIR channels driven by a common input arises in a wide range of applications. It has recently received increasing attention in the signal processing community. Much attention has been paid to identifiability-related issues of this problem. Assuming that the input to all channels is white, stationary, and infinitely long, authors of [1]–[3] studied channel identifiability conditions based on second-order statistics of the channel outputs. Assuming that the input is an unknown deterministic sequence, authors of [4] and [5] did a similar study based on a cross-relation (CR) equation in the absence of noise. An M-channel system is said to be CR identifiable if the system impulse response can be uniquely identified by the CR method [5]. In [7], channel identifiability was further analyzed based on a Fisher information (FI) matrix. An M-channel system is said to be FI identifiable if the FI matrix has nullity equal to one. An equivalence between FI identifiability and CR identifiability was established in [7].

In this paper, we study the identifiability of the M-channel FIR system in a strict sense. An M-channel FIR system is said to be strictly identifiable if the given channel outputs can only be realized by a unique system impulse response and a unique input sequence. In contrast to several existing definitions of identifiability, strict identifiability is directly based on a finite length of channel outputs instead of on certain statistics [1]–[3] or certain preprocessing [4], [5] of channel outputs. The word "strict" implies that if an M-channel system is strictly identifiable, then it must be identifiable by some method (e.g., exhaustive search); and, on the other hand, if an M-channel system is strictly not identifiable, then it can not be identifiable by any method. (It will be clear that a system can only be either strictly identifiable or strictly not identifiable. So, "stricly not identifiable" will have the same meaning as "not strictly identifiable.") Surprisingly, however, strict identifiability will be shown to be equivalent to the CR- and FI-based identifiabilities provided that the number of the output samples of each channel is no less than twice the maximum order of the FIR channels (which is a very mild condition).

Manuscript received November 4, 1994; revised September 29, 1995. This work was supported by the Australian Research Council and the Australian Cooperative Research Centre for Sensor Signals and Information Processing. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Zhi Ding.

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Publisher Item Identifier S 1053-587X(96)02406-1.

II. The M-Channel FIR System

For convenience, the M-channel system detailed in [7] is briefly reformulated in this section. For M parallel FIR channels driven by a common input sequence $s(k)$, the output of the ith channel can be written as

$$\boldsymbol{y}_i = \boldsymbol{H}_{(i)}\boldsymbol{s} \tag{2.1}$$

where $\boldsymbol{y}_i$ is the $N \times 1$ output vector of the ith channel; $\boldsymbol{H}_{(i)}$ the $N \times (N+L)$ Sylvester matrix [10] of the impulse response $h_i(k)$ of the ith channel; and s the $(N+L) \times 1$ input vector. Note that N is the total number of output samples from each channel, and L the maximum order of the M channels. Alternatively, we can write

$$\boldsymbol{y}_i = \boldsymbol{S}\boldsymbol{h}_i \tag{2.2}$$

where $\boldsymbol{S}$ is the $N \times (L+1)$ Toeplitz matrix [10] of the input sequence; and $\boldsymbol{h}_i$ the $(L+1) \times 1$ vector of the impulse response of the ith channel. Stacking all channels outputs into one vector yields, from (2.1), the following:

$$\boldsymbol{y} = \boldsymbol{H}_M\boldsymbol{s} \tag{2.3}$$

and from (2.2), the following:

$$\boldsymbol{y} = \boldsymbol{S}_M\boldsymbol{h} \tag{2.4}$$

where $\boldsymbol{y}$ is the stacked vector of $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M\}$, $\boldsymbol{H}_M$ the generalized Sylvester matrix [10] of all channels' impulse responses; $\boldsymbol{S}_M = \text{diag}\{\boldsymbol{S}, \cdots, \boldsymbol{S}\}$; and $\boldsymbol{h}$ the stacked vector of $\{\boldsymbol{h}_1, \cdots, \boldsymbol{h}_M\}$.

III. Strict Identifiability

Definition: The M-channel FIR system is said to be *strictly* identifiable from its output $\boldsymbol{y}$ if there do *not* exist $\boldsymbol{h}'$ and $\boldsymbol{s}'$ where $\boldsymbol{h}'$ is linearly independent of $\boldsymbol{h}$ *or* $\boldsymbol{s}'$ is linearly independent of $\boldsymbol{s}$ such that

$$\boldsymbol{y} = \boldsymbol{H}_M\boldsymbol{s} = \boldsymbol{H}'_M\boldsymbol{s}' \tag{3.1}$$

or equivalently

$$\boldsymbol{y} = \boldsymbol{S}_M\boldsymbol{h} = \boldsymbol{S}'_M\boldsymbol{h}' \tag{3.2}$$

where $\boldsymbol{H}'_M$ and $\boldsymbol{S}'_M$ are defined by $\boldsymbol{h}'$ and $\boldsymbol{s}'$, respectively.

It follows immediately from the definition that (i) a system that is strictly identifiable must be identifiable by some method (e.g., exhaustive search) even if not by some others, and (ii) a system that is not strictly identifiable must be strictly not identifiable (unless further constraint is imposed on the system model). This type of identifiability was also applied to the area of array processing [9]. We shall need the following three lemmas.

Lemma 1: Let the finite sequence $\{s(k), k = -L, \cdots, N-1\}$ have p (independent) modes [7]. The $N \times (L+1)$ Toeplitz matrix $\boldsymbol{S}$ has the following properties: rank$(\boldsymbol{S}) = \min(p, L+1, N)$ and range$(\boldsymbol{S}) =$ range$(\boldsymbol{M})$ for $p \leq \min(L+1, N)$ where $\boldsymbol{M}$ is the mode matrix

$$\boldsymbol{M} = \begin{bmatrix} m_1(0) & \cdots & m_p(0) \\ \vdots & \cdots & \vdots \\ m_1(N-1) & \cdots & m_p(N-1) \end{bmatrix}.$$

in which $\{m_i(k), i = 1, \cdots, p\}$ are p modes of $s(k)$.

Proof: A proof of rank$(\boldsymbol{S})$ was shown in [4] and [5]. A proof of range$(\boldsymbol{S})$ can be easily shown by verifying that each column of $\boldsymbol{S}$ can be written as a linear combination of columns of $\boldsymbol{M}$ (using a property of modes [7]).

Lemma 2 (see [1]–[3], [7], or [8]): Provided $N \geq L$, the $MN \times (N+L)$ generalized Sylvester matrix H_M has a full column rank if and only if there is no common zero among all channels.

Lemma 3 (a minor extension of [6]): Define another $MN \times (N+L)$ generalized Sylvester matrix $\boldsymbol{H}'_M$ corresponding to another impulse response vector $\boldsymbol{h}'$. If $N \geq L+1$ and $\boldsymbol{H}_M$ has its full column rank, then any of the relations range$(\boldsymbol{H}'_M) \subset$ range$(\boldsymbol{H}_M)$ and range$(\boldsymbol{H}_M) \subset$ range$(\boldsymbol{H}'_M)$ holds if and only if $\boldsymbol{h}'$ is proportional to $\boldsymbol{h}$ where $\subset$ denotes "belong to."

With the above lemmas, the necessary and sufficient conditions for strict identifiability can be established as shown below. (Recall that p is the number of modes in $s(k)$ and N is number of output samples from each channel.)

Theorem 1: The M-channel FIR system is strictly identifiable *only if* (i) all channels do not share a common zero, (ii) $p \geq L+2$, and (iii) $N \geq L+2$.

Proof: See Appendix.

Theorem 2: The M-channel FIR system is strictly identifiable *if* (i) there is no common zero among all channels, (ii) $p \geq 2L+1$, and (iii) $N \geq 3L+1$.

Proof: See Appendix.

Theorem 1 provides necessary conditions for strict identifiability and Theorem 2 gives a sufficient condition. These conditions for strict identifiability coincide with those for CR-based identifiability [4], [5] and FI-based identifiability [7] (although in [4],and [5] $p = L+1$ was not excluded from the necessary condition, and the conditions on N were not given). Before a strong connection between strict identifiability and CR- and FI-based identifiabilities will be established in Section IV, we now discuss below a more subtle issue on strict identifiability.

We can similarly define that the system *input* is strictly identifiable from a given $\boldsymbol{y}$ if there does not exist $\boldsymbol{s}'$ independent of $\boldsymbol{s}$ such that (3.1) or (3.2) holds where $\boldsymbol{h}'$ may or may not be independent of $\boldsymbol{h}$. This seems more relevant to communications where the input sequence is what we are ultimately concerned with. Furthermore, we can define that the system *impulse response* is strictly identifiable from a given $\boldsymbol{y}$ if there does not exist $\boldsymbol{h}'$ independent of $\boldsymbol{h}$ such that (3.1) or (3.2) holds where $\boldsymbol{s}'$ may or may not be independent of $\boldsymbol{s}$. From the definitions, it immediately follows that the *system* is not strictly identifiable if the *input* is not strictly identifiable or the *impulse response* is not strictly identifiable. But it is not yet apparent whether "not strictly identifiable system" implies both "not strictly identifiable input" and "not strictly identifiable impulse response." The following theorem provides a simple answer.

Theorem 3: If the M-channel FIR *system* is not strictly identifiable, both the *input* and the *impulse response* are not strictly identifiable. In other words, the *system* identifiability, the *input* identifiability, and the *impulse response* indentifiability are all equivalent to each other.

Proof: See Appendix.

IV. Equivalence to CR and FI Identifiabilities

Following [4] and [5], the M-channel FIR sytem is called CR identifiable from a given $\boldsymbol{y}$ if $\boldsymbol{h}' = \boldsymbol{h}$ is the only independent solution to the CR equation

$$\boldsymbol{Y}_M \boldsymbol{h}' = 0 \tag{4.1}$$

where $\boldsymbol{Y}_M$ is a matrix of the channel outputs as defined in [7]. To study the relationship between CR identifiability and strict identifiability, we first recall the following lemma.

Lemma 4 (see [7]): Given $\boldsymbol{H}_M$, we know a matrix $\boldsymbol{G}_M$ (detailed in [7]) such that range$(\boldsymbol{H}_M) \subset$ null$(\boldsymbol{G}_M^H)$, i.e., $\boldsymbol{G}_M^H \boldsymbol{H}_M = 0$. Furthermore, if $N \geq 2L$ (or $N \geq L+1$ for two channels) and there is no common zero among all channels, then range$(\boldsymbol{H}_M) =$ null$(\boldsymbol{G}_M^H)$.

Using this lemma, we can now establish a strong relation between CR-based identifiability and strict identifiability

Theorem 4: Provided $N \geq 2L$ (or $N \geq L+1$ for two channels), strict identifiability and CR-based identifiability are equivalent.

Proof: Recall an identity [7], as follows:

$$\boldsymbol{Y}_M \boldsymbol{h}' = \boldsymbol{G}_M'^H \boldsymbol{y} \tag{4.2}$$

where $\boldsymbol{G}'_M$ is constructed from $\boldsymbol{h}'$. First, suppose that the system is not strictly identifiable. Then (3.1) must hold for some $\boldsymbol{h}'$ independent of $\boldsymbol{h}$ (by Theorem 3). Then using (3.1) in (4.2) and applying Lemma 4, we have

$$\boldsymbol{Y}_M \boldsymbol{h}' = \boldsymbol{G}_M'^H \boldsymbol{y} = \boldsymbol{G}_M'^H \boldsymbol{H}'_M \boldsymbol{s}' = 0 \tag{4.3}$$

which implies that the system is not CR identifiable. Next, suppose that the system is not CR identifiable. Then (4.1) must have two independent solutions $\boldsymbol{h}$ and $\boldsymbol{h}''$. This in turn implies that $\boldsymbol{h}' = \boldsymbol{h} + \alpha \boldsymbol{h}''$ (where α is any complex scalar) is also a solution. Note that if $\boldsymbol{H}_M$ is not of full column rank, the system is not strictly identifiable (Theorem 1) and hence not CR identifiable. Hence, we only need to consider the case where $\boldsymbol{H}_M$ is of full column rank. In this case, we can choose a small enough α so that $\boldsymbol{h}'$ has no common zero or, equivalently, $\boldsymbol{H}'_M = \boldsymbol{H}_M + \alpha \boldsymbol{H}''_M$ has full column rank. Using this $\boldsymbol{h}'$ in (4.1) and (4.2) yields $\boldsymbol{G}_M'^H \boldsymbol{y} = 0$, which implies that $\boldsymbol{y}$ must be in null$(\boldsymbol{G}_M'^H)$. Then, by Lemma 4, $\boldsymbol{y}$ must be in range$(\boldsymbol{H}'_M)$, i.e., $\boldsymbol{y} = \boldsymbol{H}'_M \boldsymbol{s}'$ for some $\boldsymbol{s}'$ where $\boldsymbol{h}'$ is independent of $\boldsymbol{h}$, and therefore the system is not strictly identifiable. The proof is now completed.

In [7], the M-channel FIR system is defined to be FI identifiable if a Fisher information matrix has nullity equal to one. It is easy to show (using equations (3.4)–(3.6) in [7]) that the M-channel system is FI identifiable if and only if $(\boldsymbol{h}'', \boldsymbol{s}'') = (\boldsymbol{h}, -\boldsymbol{s})$ is the unique solution to $[\boldsymbol{S}_M \quad \boldsymbol{H}_M]\begin{bmatrix} \boldsymbol{h}'' \\ \boldsymbol{s}''_b \end{bmatrix} = 0$ which leads to

$$\boldsymbol{S}_M \boldsymbol{h}'' + \boldsymbol{H}_M \boldsymbol{s}'' = 0 \tag{4.4}$$

or equivalently

$$\boldsymbol{H}''_M \boldsymbol{s} + \boldsymbol{H}_M \boldsymbol{s}'' = 0. \tag{4.5}$$

Recall that a strictly identifiable system requires, by (3.1), that $(\boldsymbol{h}', \boldsymbol{s}') = (\boldsymbol{h}, \boldsymbol{s})$ is the unique solution to

$$\boldsymbol{H}_M \boldsymbol{s} - \boldsymbol{H}'_M \boldsymbol{s}' = 0. \tag{4.6}$$

A relation between the strict and FI identifiabilities, i.e., between (4.5) and (4.6), can be seen as follows. Let $\boldsymbol{s}' = \boldsymbol{s} + \epsilon \boldsymbol{s}''$ and $\boldsymbol{h}' = \boldsymbol{h} + \epsilon \boldsymbol{h}''$ where ϵ is small. Then substituting them into (4.6) and neglecting the second order terms of ϵ, we get (4.5). This means that FI-based identifiability always implies strict identifiability in a *small neighborhood* around the true system impulse response and the true input sequence. In fact, CR identifiability and FI identifiability were shown in [7] to be exactly equivalent for $N \geq 2L$ (or $N \geq L+1$ for two channels). This, together with Theorem 4, immediately implies an interesting fact that the three identifiabilities are equivalent provided $N \geq 2L$ (or $N \geq L+1$ for two channels).

Note that for $N \leq L+1$ the M-channel system is not identifiable in any sense (Theorem 1). A remaining question is: Are the three identifiabilities of a system of more than two channels equivalent for $2L > N \geq L+2$? The answer to this question is yet unknown.

V. Conclusion

We have presented the concept of strict identifiability for the M-channel FIR system and showed exact equivalence of the strict, CR, and FI identifiabilities. This implies a useful fact that provided $N \geq 2L$ (or $N \geq L+1$ for two channels), if the CR method can not yield the unique identification of an M-channel system in the absence of noise, then no method can.

Appendix

Proof of Theorem 1

To prove this theorem, it suffices to show that the system is not strictly identifiable if any of the conditions (i)–(iii) is not true.

Suppose that (i) is not true. Then by Lemma 2, $\boldsymbol{H}_M$ has a null space. This means that we can make (3.1) hold by choosing $\boldsymbol{h}' = \boldsymbol{h}$ and $\boldsymbol{s}' = \boldsymbol{s} + \boldsymbol{s}''$ where $\boldsymbol{s}''$ is from null($\boldsymbol{H}_M$) and independent of $\boldsymbol{s}$. The case where $\boldsymbol{s}$ is in null($\boldsymbol{H}_M$) and hence $\boldsymbol{y} = 0$ is of course excluded.

Suppose that (ii) is not true. In this case, either $p < L+1$ or $p = L+1$. If $p < L+1$, then by Lemma 1, $\boldsymbol{S}$ (and hence $\boldsymbol{S}_M$) has a null space. This means that we can make (3.2) hold by choosing $\boldsymbol{s}' = \boldsymbol{s}$ and $\boldsymbol{h}' = \boldsymbol{h} + \boldsymbol{h}''$ where $\boldsymbol{h}''$ is from null($\boldsymbol{S}_M$) and independent of $\boldsymbol{h}$. Now we consider the case $p = L+1$. In this case, if $N < L+1$, $\boldsymbol{S}$ has a null space and hence we can make (3.2) hold for some $\boldsymbol{h}'$ independent of $\boldsymbol{h}$. Otherwise, if $p = L+1$ and $N \geq L+1$, we can construct a sequence $s'(k)$

$$s'(k) = \sum_{i=1}^{p} c_i' m_i(k) \tag{A.1}$$

such that $\boldsymbol{s}'$ is independent of $\boldsymbol{s}$. For this choice of $\boldsymbol{s}'$, we have from Lemma 1 range($\boldsymbol{S}'$) = range($\boldsymbol{M}$) = range($\boldsymbol{S}$). Therefore, for any $\boldsymbol{h}$ there exists a $\boldsymbol{h}'$ such that (3.2) holds.

Finally, suppose that (iii) is not true. Then S_M either has a null space (i.e., when $N < L+1$ or $p < L+1$) or is nonsingular (i.e., when $N = L+1$ and $p \geq L+1$) with its range being the complex space of dimension $M(L+1)$. In the former case, we can make (3.2) hold by choosing $\boldsymbol{s}' = \boldsymbol{s}$ and $\boldsymbol{h}'$ independent of $\boldsymbol{h}$. In the latter case, we can make (3.2) hold by choosing a $\boldsymbol{s}'$, which is independent of $\boldsymbol{s}$ and makes $\boldsymbol{S}'$ nonsingular. The proof is now completed.

Proof of Theorem 2

Define a $(N-L) \times (2L+1)$ input matrix

$$\boldsymbol{S}_b = \begin{bmatrix} s(-L) & \cdots & s(L) \\ \vdots & \vdots & \vdots \\ s(N-2L-1) & \cdots & s(N-1) \end{bmatrix}$$

a $(N-L) \times (L+1)$ output matrix for the ith channel

$$\boldsymbol{Y}_{(i)} = \begin{bmatrix} y_i(L) & \cdots & y_i(0) \\ \vdots & \vdots & \vdots \\ y_i(N-1) & \cdots & y_i(N-L-1) \end{bmatrix}$$

and a $(2L+1) \times (L+1)$ impulse response matrix for the ith channel

$$\boldsymbol{H}_{(i),b} = \begin{bmatrix} & & h_i(L) \\ & \vdots & \vdots \\ h_i(L) & \vdots & h_i(0) \\ \vdots & \vdots & \\ h_i(0) & & \end{bmatrix}.$$

Further define $\boldsymbol{Y}_b = [\boldsymbol{Y}_{(1)} \ \boldsymbol{Y}_{(2)} \ \cdots \ \boldsymbol{Y}_{(M)}]$ and $\boldsymbol{H}_b = [\boldsymbol{H}_{(1),b} \ \boldsymbol{H}_{(2),b} \ \cdots \ \boldsymbol{H}_{(M),b}]$. Note that $\boldsymbol{H}_b$ is also a generalized Sylvester matrix, and its transpose has the identical structure as $\boldsymbol{H}_M$ except the dimension. Then (2.1) or (2.2) implies

$$\boldsymbol{Y}_b = \boldsymbol{S}_b \boldsymbol{H}_b. \tag{A.2}$$

It follows obviously that the system is strictly identifiable if and only if there do not exist $\boldsymbol{h}'$ and $\boldsymbol{s}'$, where $\boldsymbol{h}'$ is independent of $\boldsymbol{h}$ or $\boldsymbol{s}'$ is independent of $\boldsymbol{s}$ such that $\boldsymbol{Y}_b = \boldsymbol{S}_b \boldsymbol{H}_b = \boldsymbol{S}_b' \boldsymbol{H}_b'$ or, equivalently

$$[\boldsymbol{S}_b - \boldsymbol{S}_b'] \begin{bmatrix} \boldsymbol{H}_b \\ \boldsymbol{H}_b' \end{bmatrix} = 0 \tag{A.3}$$

where $\boldsymbol{S}_b'$ and $\boldsymbol{H}_b'$ are defined by $\boldsymbol{s}'$ and $\boldsymbol{h}'$, respectively. For (A.3) to hold, a necessary condition is

$$\text{nullity}[\boldsymbol{S}_b \quad \boldsymbol{S}_b'] \geq \text{rank} \begin{bmatrix} \boldsymbol{H}_b \\ \boldsymbol{H}_b' \end{bmatrix}. \tag{A.4}$$

To prove the theorem, it now suffices to show that under the condition of the theorem, (A.4) does not hold if $\boldsymbol{s}'$ is independent of $\boldsymbol{s}$ or $\boldsymbol{h}'$ is independent of $\boldsymbol{h}$. By an obvious variant of Lemma 1, rank($\boldsymbol{S}_b$) $= 2L+1$ if $s(k)$ has no less than $2L+1$ modes and $\boldsymbol{S}_b$ has no less than $2L+1$ rows (i.e., $N \geq 3L+1$). Similarly, by an obvious variant of Lemma 2, rank($\boldsymbol{H}_b$) $= 2L+1$ if there is no common zero among all channels. The above implies that

$$\begin{aligned} \text{nullity}[\boldsymbol{S}_b \quad \boldsymbol{S}_b'] &= 4L+2 - \text{rank}[\boldsymbol{S}_b \quad \boldsymbol{S}_b'] \leq 4L+2 \\ &\quad - \text{rank}(\boldsymbol{S}_b) = 2L+1 \end{aligned} \tag{A.5}$$

and

$$\text{rank} \begin{bmatrix} \boldsymbol{H}_b \\ \boldsymbol{H}_b' \end{bmatrix} \geq \text{rank}(\boldsymbol{H}_b) \geq 2L+1. \tag{A.6}$$

Suppose that $\boldsymbol{h}'$ is indepedent of $\boldsymbol{h}$. Then by an obvious variant of Lemma 3, the row space of $\boldsymbol{H}_b'$ does not belong to that of $\boldsymbol{H}_b$ and, hence, $\begin{bmatrix} \boldsymbol{H}_b \\ \boldsymbol{H}_b' \end{bmatrix} \geq \text{rank}(\boldsymbol{H}_b) + 1 \geq 2L+2$. This, together with (A.5), implies that (A.4) does not hold. Now, suppose that $\boldsymbol{s}'$ is independent of $\boldsymbol{s}$. In this case, either $\boldsymbol{S}_b' \neq \boldsymbol{S}_b \boldsymbol{T}$ or $\boldsymbol{S}_b' = \boldsymbol{S}_b \boldsymbol{T}$ where $\boldsymbol{T}$ is a $(2L+1) \times (2L+1)$ matrix. If $\boldsymbol{S}_b' \neq \boldsymbol{S}_b \boldsymbol{T}$, then

$$\begin{aligned} \text{nullity}[\boldsymbol{S}_b \quad \boldsymbol{S}_b'] &= 4L+2 - \text{rank}[\boldsymbol{S}_b \quad \boldsymbol{S}_b'] \leq 4L+2 \\ &\quad - \text{rank}(\boldsymbol{S}_b) - 1 = 2L. \end{aligned}$$

This, together with (A.6), implies that (A.4) does not hold. If $\boldsymbol{S}_b' = \boldsymbol{S}_b \boldsymbol{T}$, then (A.3) implies

$$\boldsymbol{H}_b = \boldsymbol{T} \boldsymbol{H}_b'. \tag{A.7}$$

By an obvious variant of Lemma 3, (A.7) holds only if $\boldsymbol{T}$ is a scaled version of the identity matrix. This contradicts the assumption that $\boldsymbol{s}'$ is independent of $\boldsymbol{s}$. The proof is completed.

Proof of Theorem 3

It suffices to show that (a) if a given $\boldsymbol{y} (= \boldsymbol{H}_M \boldsymbol{s})$ implies a unique $\boldsymbol{h}$, it must also imply a unique $\boldsymbol{s}$, and (b) if a given $\boldsymbol{y} (= \boldsymbol{S}_M \boldsymbol{h})$ implies a unique $\boldsymbol{s}$, it must also imply a unique $\boldsymbol{h}$.

Case (a) is clearly true if $\boldsymbol{H}_M$ has full column rank, or equivalently, all channels share no common zero. But if all channels share a common zero, say z_0, a given $\boldsymbol{y}$ cannot imply a unique $\boldsymbol{h}$ for the following reason. Since $h_i(z)$ can be factorized as $h_i'(z)h_0(z)$, we can write $\boldsymbol{H}_{(i)} = \boldsymbol{H}'_{(i)}\boldsymbol{H}_0$ and hence $\boldsymbol{H}_M = \boldsymbol{H}'_M\boldsymbol{H}_0$. Therefore, we can construct $\boldsymbol{y} = \boldsymbol{H}_M\boldsymbol{s} = \boldsymbol{H}'_M\boldsymbol{s}'$ where $\boldsymbol{s}' = \boldsymbol{H}_0\boldsymbol{s}$ and the corresponding $\boldsymbol{h}'$ is independent of $\boldsymbol{h}$. Case (b) is also clearly true if $\boldsymbol{S}_M$ has a full column rank, or equivalently, $p \geq L+1$ and $N \geq L+1$. However, if $p < L+1$ or $N < L+1$, a given $\boldsymbol{y}$ cannot imply a unique $\boldsymbol{s}$ because we can construct $\boldsymbol{y} = \boldsymbol{S}_M\boldsymbol{h} = \boldsymbol{S}'_M\boldsymbol{h}'$ where $\boldsymbol{s}'$ is independent of $\boldsymbol{s}$, as follows. If $\boldsymbol{S}_M$ has a full row rank, then any $\boldsymbol{S}'_M$ of full row rank and a proper choice of $\boldsymbol{h}'$ will make the above equation hold. If $\boldsymbol{S}_M$ does not have a full row rank, i.e., by Lemma 1, $\mathrm{rank}(\boldsymbol{S}_M) = pM < \min(NM, (L+1)M)$, then one can construct $\boldsymbol{y} = \boldsymbol{S}_M\boldsymbol{h} = (S_M + S''_M)(\boldsymbol{h}+\boldsymbol{h}'')$ where $\boldsymbol{h}''$ has p common zeros defined by the p modes in $\boldsymbol{s}$ (making $\boldsymbol{S}_M\boldsymbol{h}'' = 0$), and $\boldsymbol{s}''$, associated with $\boldsymbol{S}''_M$, consists of the modes defined by the common zeros of $\boldsymbol{h}+\boldsymbol{h}''$ (making $\boldsymbol{S}''_M(\boldsymbol{h}+\boldsymbol{h}'') = 0$). Note that a common zero of $\boldsymbol{h}+\boldsymbol{h}''$ can be constructed as follows. The transfer function of the ith channel corresponding to $\boldsymbol{h}+\boldsymbol{h}''$ is $H_i'(z) = H_i(z) + a_iH_i''(z)$ where $H_i''(z)$ has p zeros that are independent of i. For any z_0 for which $H_i''(z_0) \neq 0$, there is a_i such that $H_i'(z_0) = 0$. The proof is completed.

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Identification of 2-D Noncausal Gauss–Markov Random Fields

R. Cusani and E. Baccarelli

***Abstract*—An original procedure for estimating the model parameters of discrete-index 2-D noncausal Gauss-Markov random fields (GMRF's) from noisy observations is proposed, valid for both finite and infinite lattices and for any kind of boundary conditions. Starting from a suitable "local" representation of the GMRF and taking into account the symmetry property of so-called field potentials, a linear equation set relating the model parameters to the 2-D autocorrelation function (known or estimated) of the observed field is derived. Its solution gives the parameter estimates of the GMRF together with the estimate of the (possibly unknown) variance of the observation noise.**

I. Representation and Some Properties of 2-D Noncausal GMRF

Parameter identification of multidimensional noncausal Markov random fields is an important paradigm in multidimensional signal processing and modeling, and the solutions to this problem are employed in many applicative image processing areas [1], [2].

In order to derive a representation for a 2-D noncausal GMRF, we introduce the following:

—A rectangular lattice I (finite or infinite) in the Euclidean space R^2 defined as the set of sites $I \equiv \{\underline{s} \equiv (s_1, s_2){:}\ -N_0 \leq s_1 \leq N_1, -M_0 \leq s_2 \leq M_1\}$,with s_1, s_2 integers and with N_0, N_1, M_0, M_1 nonnegative assigned integers (eventually, infinite)

—A noncausal "neighborhood system" $\eta(d) \equiv \{\underline{r} \equiv (r_1, r_2){:}\, 0 < (r_1^2 + r_2^2) \leq d\}$,with r_1, r_2 integers based on the Euclidean distance measure [1], [3], [4] also called "support region" [2]. The adopted definition for the neighborhood system $\eta(d)$ is largely employed to define noncausal Markov fields [5, Section III]; for $d = 1, 2, 4, 5, 8, \ldots$ GMRF's of order $1, 2, 3, 4, 5, \ldots$ are obtained. The corresponding support regions are symmetric (i.e., if $\underline{r} \in \eta(d)$ then $-\underline{r} \in \eta(d)$) and constituted by an even number of points, denoted as $2L(d)$

—The set of "internal points" of the lattice I with respect to (w.r.t.) $\eta(d)$, defined as $I^\circ(\eta(d)) \equiv \{\underline{s} \in I{:}\, \underline{s} + \underline{r} \in I$ for every $r \in \eta(d)\}$ where the operation of sum between sites is defined as $\underline{s} + \underline{r} \equiv (s_1 + r_1, s_2 + r_2)$

—The set of 'boundary points' of the lattice I w.r.t. $\eta(d)$ denoted by $\partial I(\eta(d)) \equiv I \setminus I^\circ(\eta(d))$.

A discrete-index 2-D zero-mean Gaussian random process $\{X(\underline{s}) \in R^1, \underline{s} \in I\}$ defined on the lattice I constitutes a noncausal GMRF w.r.t. the assigned support region $\eta(d)$ if $E\{X(\underline{s}) \mid X(\underline{t}), \underline{t} \in I, t \neq \underline{s}\} = E\{X(\underline{s}) \mid X(\underline{s}+\underline{r}), \underline{r} \in \eta(d)\}, \forall \underline{s} \in I^\circ(\eta(d))$ [5]. Therefore, an *homogeneous* GMRF admits the "innovations representation" [2], [5] as follows:

$$x(\underline{s}) = \sum_{\underline{r} \in \eta(d)} \phi(\underline{r}) x(\underline{s}+\underline{r}) + u(\underline{s}), \quad \underline{s} \in I^\circ(\eta(d)) \qquad (1)$$

also denoted as "minimum variance" or "nearest neighbor" representation, where the 2-D "innovations process" $\{U(\underline{s}) \in R^1, \underline{s} \in$

Manuscript received August 4, 1994; revised September 11, 1995. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Michael Zervakis.

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Publisher Item Identifier S 1053-587X(96)02404-8.